Statistical Field Theory for Latecomers

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A Course in 20 lectures held in the spring semester of 2015 for the PhD students in Physics at the University Sapienza

- **Part 1: Statistical Mechanics** (10 lectures, 20 hrs)

  1. *Ensembles - Energy, entropy, free energy*
     - Microcanonical ensemble
     - Maximum entropy principle
     - Canonical ensemble
     - Helmoltz free energy
     - A tale of many entropies
     - Thermodynamic temperature
  2. *Gibbs free energy and susceptibility*
     - Magnetization m and magnetic field h
     - Legendre transform
     - Gibbs and Helmoltz free energy: $g_{\mathrm{G}}(m)$ vs $f_{\mathrm{H}}(h)$
     - Susceptibility χ: fluctuation and curvature of the free energy
     - The generalized (nonconvex) Gibbs free energy $g(m)$
  3. *Correlation function*
     - Connected correlation function $G(r)$
     - The role of the fluctuations
     - Correlation length ξ
     - Susceptibility and the space integral of $G(r)$
     - Relationship between χ and ξ
  4. *Correlation function w/out ensemble averages*
     - What happens when you have real data
     - Space-averaged correlation function $G(r)$
     - How to connect the correlation function
     - The effect of the sum rule
     - Proxies of the correlation length ξ
     - Long-ranged vs short-ranged $G(r)$
  5. *Phase transitions*
     - Competition between energy and entropy
     - Ising model and the mechanism of imitation
     - Mean-field solution of the ferromagnetic p-spin model
     - Explicit calculation of the generalized Gibbs free energy $g(m)$
     - Second order phase transition ($p = 2$) and the transition temperature T_c
     - First order phase transition ($p = 3$) and the spinodal temperature T_d
  6. *Problems & Solutions class*
     - Saddle point method
     - Proof that $g''(m)$ can change sign also for $N < \infty$
     - Why $\chi < \infty$ for $N < \infty$ even when $g''(m) = 0$?
     - General form of $\partial m(T)/\partial T$ and its divergence at T_c
  7. *Form of the Gibbs free energy below T_c*
     - What happens to $g_{\mathrm{G}}(m)$ when $g''(m) < 0$
     - The $h \to 0$ vs $N \to \infty$ limit
     - Spontaneous symmetry breaking
     - The function $m(h)$
     - Maxwell's construction
     - Phase separation
     - Is χ finite or not in a first order phase transition?
  8. *Metastability and classic nucleation theory*
     - Local minima of the generalized Gibbs free energy $g(m)$
     - Droplet argument: surface tension resistance vs free energy drive
     - Expression of the free energy barrier
     - Role of the dimension and of finite size
     - The special case of mean field
  9. *Dynamics I*
     - Langevin equation
     - Green functions method
     - Explicit solution for a free particle (Brownian motion)
     - Einsten's relation between friction and noise
     - Overdamped limit and the rescaling of time
     - Explicit solution for a particle in a harmonic potential
       * ballistic regime
       * diffusive regime
       * saturation
  10. *Dynamics II*
      - Dynamical propagator and response
      - Dynamical correlation function
      - General relations between response and correlation
      - Gaussian dynamics in the scalar and vectorial case
      - Comparison between dynamics and statics

- **Part 2: Field Theory** (10 lectures, 20 hrs)

1. *Landau-Ginzburg model*
   - Role of scale invariance
   - Coarse graining: from σ_i to $\varphi(x)$
   - Entropy contribution
   - Energy contribution
   - The change of sign of the bare mass $\mu^2(T)$
   - The Landau-Ginzburg $\lambda\varphi^4$ Hamiltonian
2. *Landau approximation (LA)*
   - The simplest solution of the Landau-Ginzburg model: $\varphi(x) = \varphi_0$
   - Physical meaning of the LA: $\xi \ll v^{1/d}$
   - LA and mean-field Gibbs free energy
   - Critical temperature in the LA
   - Critical exponents in the LA
3. *Gaussian field theory I*
   - Meaning of the Gaussian case
   - Modes separation and exact solution
   - The Gaussian propagator $G_0(k) = \frac{1}{k^2+\mu^2}$
   - $G(r)$ and role of the cutoff Λ
   - Changing the cutoff Λ: a foreshadow of renormalization
   - Gaussian and Landau critical exponents from dimensional analysis
4. *Gaussian field theory II*
   - A physical realization of the Gaussian case: anchored oscillators chain
   - Symmetry braking and zero mode
   - How can the correlation be inversely proportional to the interaction?
   - Dynamics of the Gaussian field theory
5. *Diagrammatic expansion*
   - Wick theorem
   - Feynman diagrams
   - Expansion of $G(k)$: bubble, saturn and cactus diagrams
   - Vacuum fluctuations
   - 1PI and amputated diagrams
   - Self energy and Dyson equation
   - Vertex function $\Gamma^{(2)}(k)$ and susceptibility
   - Loop expansion vs λ-expansion
6. *Renormalization I*
   - General idea: arbitrariness of the cutoff Λ
   - Mass renormalization at one loop
   - Mass renormalization at two loops
   - Insensitivity to the cutoff Λ at two loops
   - Substituting $\mu^2 \to m^2$ in all diagrams
7. *Renormalization II*
   - Field renormalization
   - Coupling constant renormalization
   - When is a theory renormalizable?
   - Ginzburg criterion: $d > 4$ vs $d < 4$
8. *The Renormalization Group I*
   - Brief historical overview
   - Real space renormalization: blocking
   - Fixed points and critical manifold
   - Momentum shell renormalization: Gaussian case
     * The $\{\varphi_>(k),\, \varphi_<(k)\}$ split
     * Recursive relations and β-function
     * What is special in the Gaussian case?
     * Instability of $\lambda = 0$ under renormalization
9. *The Renormalization Group II*
   - Momentum shell renormalization at one loop
     * Why $\lambda\varphi^4$ is different: the coupling $\varphi_>(k) \cdot \varphi_<(k)$
     * Recursive equation for the mass μ^2 and the coupling constant λ
     * The key idea of the ϵ-expansion
     * The Wilson-Fisher fixed point
     * The flow of the coupling constant λ: $d > 4$ vs $d < 4$
   - Callan-Symanzik (CS) equations
     * CS equations in bare space
     * Comparison of the β-function in CS and in momentum shell: $\int_{\Lambda/b}^{\Lambda}$ vs $\int_0^{\Lambda/b}$
     * CS in renormalized space
10. *Continuous symmetry breaking*
   - Discrete vs continuous symmetry
   - The $\lambda(\vec{\varphi}\cdot\vec{\varphi})^2$ model below T_c and the mexican hat paradigm
   - The massless Goldstone mode
   - Derivation of Goldstone theorem from the Ward identities
   - Mermin-Wagner theorem and the role of spin waves
   - $\chi = G(k)|_{k=0}$ vs $\langle\delta\varphi^2\rangle = G(r)|_{r=0}$
   - The effect of the vertex $\varphi_\parallel\varphi_\perp\varphi_\perp$
   - An unexpected result: $\chi_\parallel \sim \chi_\perp^{\epsilon/2} \to \infty$
   - Physical meaning of the divergence of the longitudinal susceptibility
   - Where is the massive particle?