Cattedra Enrico Fermi 2015-2016

La teoria delle stringhe:
l'ultima rivoluzione in fisica?

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Lezione # 5.2: 10.12.2015

Un primo sguardo allo spettro del DRM:
Temperatura di Hagedorn

Solving the planar duality constraints

Chan's form for the N-point function (1968)

Chan's form for the N-point function is the most direct generalization of the integral form of the Beta function (now called B_4) for 4 spinless particles:

$$B_4\left(-\alpha(s), -\alpha(t)\right) = \int_0^1 dx\ x^{-1-\alpha(s)}(1-x)^{-1-\alpha(t)}$$

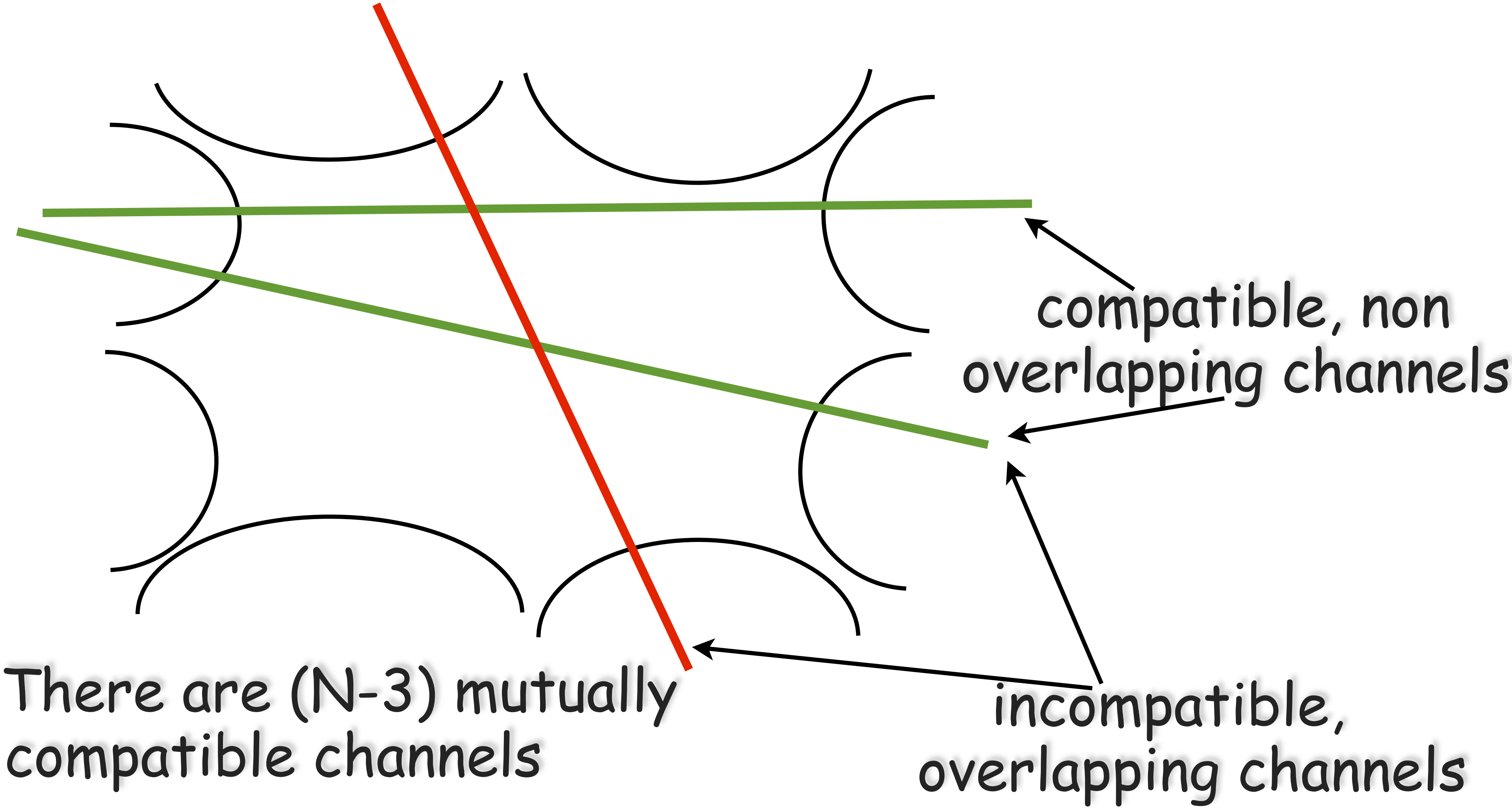
$$B_N = \int_0^1 \prod_{ij} du_{ij} u_{ij}^{-1-\alpha(s_{ij})} \delta(\dots)$$

where the δ-functions eliminate all but (N-3) integration variables (the maximal number of compatible poles) via the constraints:

$$1 - u_P = \prod_{\bar{P}} u_{\bar{P}}$$

where the product extends to all channels overlapping with P.

Planar duality is very natural from a duality diagram viewpoint

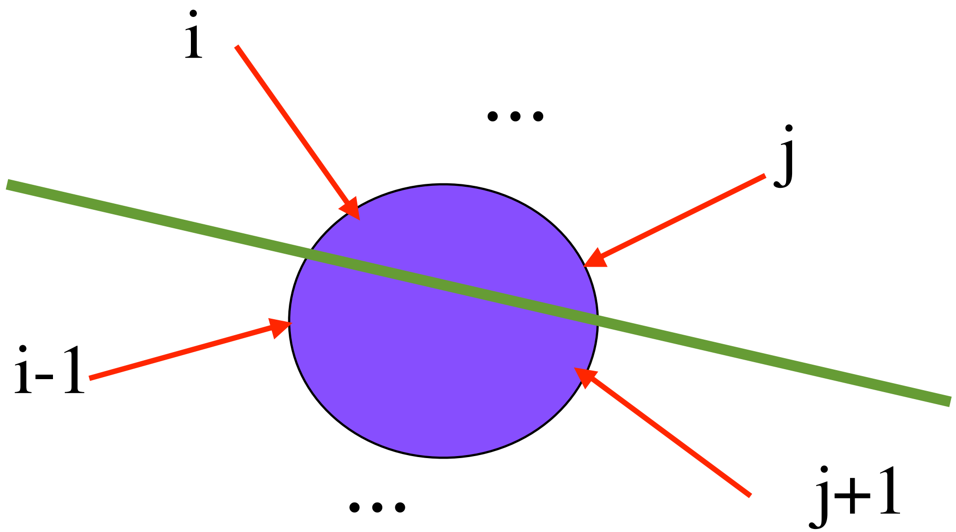


Koba-Nielsen form (for special case α(0) =1)

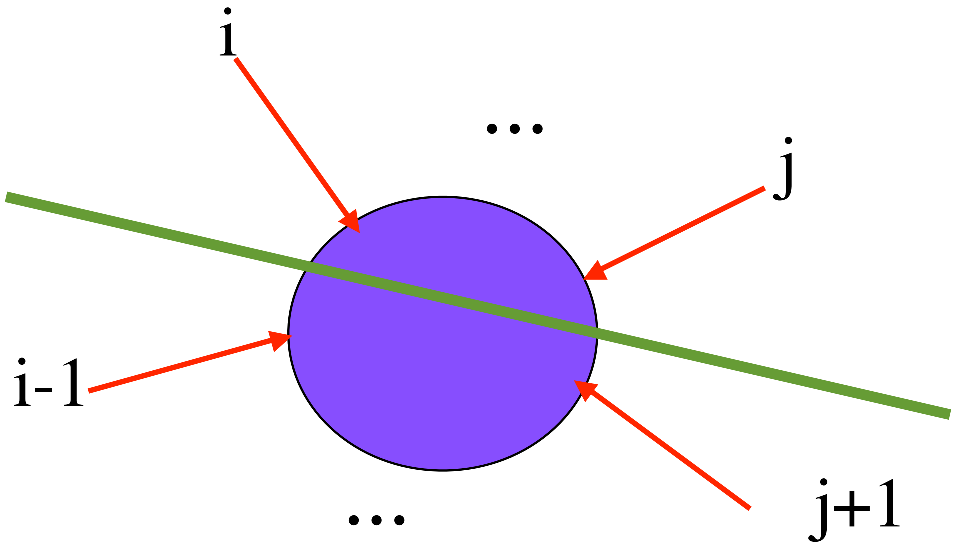
The most elegant (and useful) solution to the constraints was given by Z. Koba & H. Nielsen (1968).

Their construction is as follows:

Associate with each external particle a real variable z_i ($i = 1, 2, \ldots N$) and to each planar channel a particular anharmonic ratio of the z's:



$$\equiv \quad \left(z_i, z_{i-1}, z_j, z_{j+1}\right) \equiv \frac{\left(z_i - z_j\right)\left(z_{i-1} - z_{j+1}\right)}{\left(z_{i-1} - z_j\right)\left(z_i - z_{j+1}\right)}$$



$$\equiv \frac{(z_i - z_j)(z_{i-1} - z_{j+1})}{(z_{i-1} - z_j)(z_i - z_{j+1})}$$

$(z_i, z_{i-1}, z_j, z_{j+1})$

B_N is then given by (w/ a,b,c chosen arbitrarily):

$$B_N = \int_{-\infty}^{+\infty} dV(z) \prod_{i,j} (z_i, z_{i-1}, z_j, z_{j+1})^{-1-\alpha(s_{ij})}$$

$$dV(z) = \frac{\prod dz_i \theta(z_i - z_{i+1})}{\prod (z_i - z_{i+2}) dV_{abc}}$$

$$dV_{abc} = \frac{dz_a dz_b dz_c}{(z_b - z_a)(z_c - z_b)(z_a - z_c)}$$

$$
\begin{aligned}
B_N &= \int_{-\infty}^{+\infty} dV(z) \prod_{i,j} (z_i, z_{i-1}, z_j, z_{j+1})^{-1-\alpha(s_{ij})} \\
dV(z) &= \frac{\prod dz_i \theta(z_i - z_{i+1})}{\prod (z_i - z_{i+2}) dV_{abc}} \\
dV_{abc} &= \frac{dz_a dz_b dz_c}{(z_b - z_a)(z_c - z_b)(z_a - z_c)}
\end{aligned}
$$

Duality constraints automatically satisfied:

Integrand and integration measure are invariant under projective O(2,1) transformations:

$$
z_i \to \frac{\alpha z_i + \beta}{\gamma z_i + \delta} \; ; \; \alpha\delta - \beta\gamma = 1
$$

N.B. Without dividing by dV_{abc} one would get infinity.

3 z's can be fixed arbitrarily leaving (N-3) integration variables.

$$B_N = \int_{-\infty}^{+\infty} dV(z) \prod_{i,j} (z_i, z_{i-1}, z_j, z_{j+1})^{-1-\alpha(s_{ij})}$$

$$dV(z) = \frac{\prod dz_i \theta(z_i - z_{i+1})}{\prod (z_i - z_{i+2}) dV_{abc}}$$

$$dV_{abc} = \frac{dz_a dz_b dz_c}{(z_b - z_a)(z_c - z_b)(z_a - z_c)}$$

Using relations such as:

$$\gamma_{ij} = \alpha(s_{ij}) + \alpha(s_{i+1,j-1}) - \alpha(s_{i+1,j}) - \alpha(s_{i,j-1}) = -2\alpha' p_i p_j$$

we collect all the factors that contain a given $(z_i\text{-}z_j)$ and obtain (for $\alpha(0) = 1!$) the standard KN form:

$$B_N = \int_{-\infty}^{+\infty} \frac{\prod dz_i \theta(z_i - z_{i+1})}{dV_{abc}} \prod_{j>i} (z_i - z_j)^{2\alpha' p_i \cdot p_j}$$

$$B_N = \int_{-\infty}^{+\infty} \frac{\prod dz_i \theta(z_i - z_{i+1})}{dV_{abc}} \prod_{j>i} (z_i - z_j)^{2\alpha' p_i \cdot p_j}$$

Note that the integrand is now independent of the cyclic ordering of the external lines. This only appears in the integration measure through the ordering of the z's (again only for $\alpha(0) = 1$).

A convenient choice for the 3 fixed z's is:

$$z_a = z_1 = \infty \ ; \ z_b = z_2 = 1 \ ; \ z_c = z_N = 0$$

$$B_N = \prod_3^{N-1} \left[\int_0^1 dz_i \theta(z_i - z_{i+1}) \right] \prod_{i=2}^{N-1} \prod_{j=i+1}^{N} (z_i - z_j)^{2\alpha' p_i \cdot p_j}$$

$$B_N = \prod_3^{N-1} \left[\int_0^1 dz_i \theta(z_i - z_{i+1})\right] \prod_{i=2}^{N-1} \prod_{j=i+1}^{N} (z_i - z_j)^{2\alpha' p_i \cdot p_j}$$

This was the starting point of the original study of the spectrum (FV & BM, 1969). The simplest way to describe it is by introducing (FGV, N, 1969) an operator formalism:

$$[q_\mu \ , \ p_\nu] = i\eta_{\mu\nu} \ , \quad [a_{n,\mu} \ , \ a^\dagger_{m,\nu}] = \delta_{n,m}\eta_{\mu\nu} \ , \quad \eta_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$$

$$(n = 1, 2, \ldots \ ; \ \mu = 0, 1, 2, \ldots D - 1)$$

They look like our previous harmonic-oscillator operators in D dimensions (that we called N, sorry) except that:

1. For each dimension there is also an infinity of oscillators (corresponding to higher harmonics, as we shall see);
2. One set of them (corresponding to time) has the "wrong" sign: the price for (manifest) relativistic invariance!

One can show (left below for those interested) that a sufficient set of states consists of the eigenstates of momentum and of the occupation numbers of those harmonic oscillators i.e.

$$|N_{n,\mu}, k\rangle \sim \prod_{n,\mu} \left(a^{\dagger}_{n,\mu}\right)^{N_{n,\mu}} e^{iqk}|0\rangle \quad ; \quad a_{n,\mu}|0\rangle = p_{\mu}|0\rangle = 0$$

$$-\alpha' k^2 = \alpha' M^2 = -1 + \sum_{n,\mu} n \, a^{\dagger}_{n,\mu} \, a^{\mu}_{n}$$

(relativistic analog of E of h.o.)

Because of the "wrong" sign of the time-like c.r., states created by an odd number of time-like operators are ghosts. Was the DRM doomed? One (tiny?) hope remained: all those states were sufficient but perhaps only a (ghost-free?) subset was necessary.

Next week we will see that the ghost killing program indeed works but, once more, at a price!

Proof of factorization (for those interested)

We shall now rewrite the KN form of B_N using our operators. Two essential ingredients are:

1) a "field operator" $Q_\mu(z)$ and

2) a "vertex operator" V(z, k)

$$Q_\mu(z) = Q_\mu^{(0)}(z) + Q_\mu^{(+)}(z) + Q_\mu^{(-)}(z) \quad ; \quad Q_\mu^{(0)}(z) = q_\mu - 2i\alpha' p_\mu log z$$

$$Q_\mu^{(+)}(z) = i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{a_{n,\mu}}{\sqrt{n}}z^{-n} \quad ; \quad Q_\mu^{(-)}(z) = -i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{a^\dagger_{n,\mu}}{\sqrt{n}}z^{n}$$

$$V(z,k) =: e^{ik\cdot Q(z)} : \equiv e^{ik\cdot Q^{(-)}(z)}\, e^{ik\cdot q}\, e^{2\alpha' k\cdot p log z}\, e^{ik\cdot Q^{(+)}(z)}$$

They satisfy the following operator identities:

$$Q_\mu(z) = Q_\mu^{(0)}(z) + Q_\mu^{(+)}(z) + Q_\mu^{(-)}(z) \quad ; \quad Q_\mu^{(0)}(z) = q_\mu - 2i\alpha' p_\mu log z$$

$$Q_\mu^{(+)}(z) = i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{a_{n,\mu}}{\sqrt{n}} z^{-n} \quad ; \quad Q_\mu^{(-)}(z) = -i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{a_{n,\mu}^\dagger}{\sqrt{n}} z^{n}$$

$$V(z,k) =: e^{ik\cdot Q(z)} : \;\equiv\; e^{ik\cdot Q^{(-)}(z)}\, e^{ik\cdot q}\, e^{2\alpha' k\cdot p log z}\, e^{ik\cdot Q^{(+)}(z)}$$

$$[Q_\mu^{(+)}(z), Q_\nu^{(-)}(w)] = -2\alpha' log\left(1 - \frac{w}{z}\right)\eta_{\mu\nu}$$

$$V(z,k)V(w,k') =: V(z,k)V(w,k') : (z-w)^{2\alpha' k\cdot k'}$$

leading easily to:

$$\langle 0| \prod_{i=1}^{N} V(z_i, p_i)|0\rangle = (2\pi)^D \delta^{(D)}(\sum p_i) \prod_{i>j} (z_i - z_j)^{2\alpha' p_i\cdot p_j}$$

Consequently, recalling

$$B_N = \int_{-\infty}^{+\infty} \frac{\prod dz_i \theta(z_i - z_{i+1})}{dV_{abc}} \prod_{j>i} (z_i - z_j)^{2\alpha' p_i \cdot p_j}$$

we have the elegant result:

$$(2\pi)^D \delta^{(D)}(\sum p_i) B_N = \int_{-\infty}^{+\infty} \frac{\prod dz_i \theta(z_i - z_{i+1})}{dV_{abc}} \langle 0| \prod_{i=1}^{N} V(z_i, p_i) |0\rangle$$

This looks already nicely factorized. To complete the proof we use the fact that the operator

$$L_0 = \alpha' p^2 + \sum_{n,\mu} n \; a^\dagger_{n,\mu} \; a^\mu_n$$

acts on Q as z d/dz Q giving:

$$V(z,k) = z^{L_0 - \alpha' k^2} V(1,k) z^{-L_0} = z^{L_0 - 1} V(1,k) z^{-L_0}$$

Using this repeatedly and performing the explicit integrals on z_{i+1}/z_i we finally arrive at the desired fully factorized form:

$$
\begin{aligned}
(2\pi)^D \delta^{(D)}(\sum p_i) B_N &= \langle p_1 | V(1, p_2)\ D\ V(1, p_3)\ D\ V(1, p_4)\ D \ldots D\ V(1, p_{N-1}) | p_N \rangle \\
D &= \frac{1}{L_0 - 1}
\end{aligned}
$$

In order to factorize this amplitude it's enough to introduce a complete set of harmonic oscillator states before and after a given "propagator" D. This will provide a pole at:

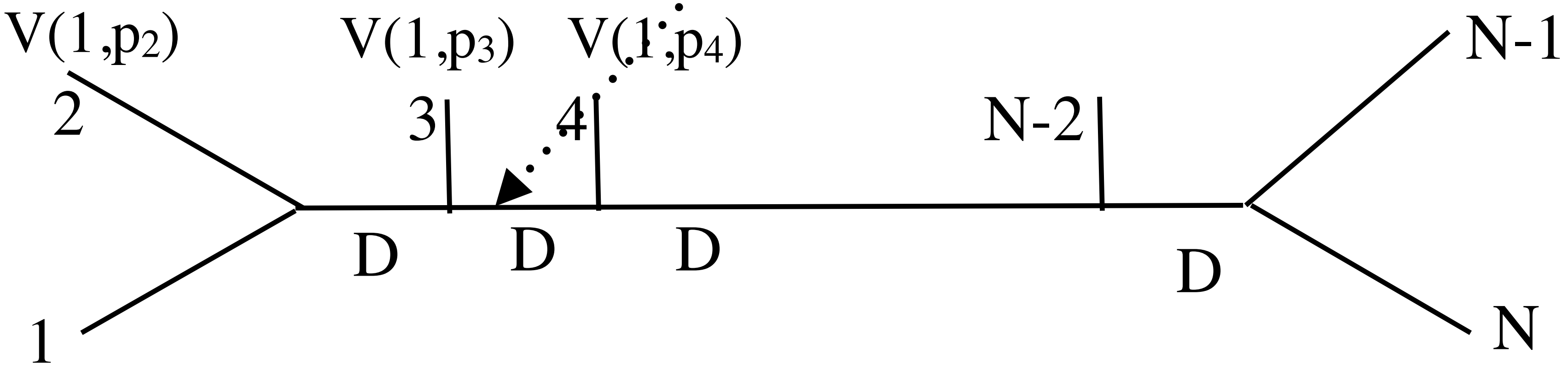
$$
L_0 = 1 \Rightarrow -\alpha' p^2 = \alpha' M^2 = -1 + \sum_{n,\mu} n\ a^{\dagger}_{n,\mu}\ a^{\mu}_{n} = -1 + \sum_{n,\mu} n N_{n,\mu}
$$

In pictures

$$(2\pi)^D\delta^{(D)}(\sum p_i)B_N = \langle p_1|V(1,p_2)\;D\;V(1,p_3)\;D\;V(1,p_4)\;D\ldots D\;V(1,p_{N-1})|p_N\rangle$$

$$D = \frac{1}{L_0-1}$$

$$1=\sum\int dk|N_{n,\mu},k\rangle\langle N_{n,\mu},k|$$



$$L_0=1\Rightarrow -\alpha' p^2=\alpha' M^2=-1+\sum_{n,\mu} n\; a^\dagger_{n,\mu}\; a^\mu_n = -1+\sum_{n,\mu} nN_{n,\mu}$$

(Over?) counting states

- The mass^2 condition:

$$L_0 = 1 \Rightarrow \alpha' M^2 + 1 = \sum_{n,\mu} n\ a^\dagger_{n,\mu}\ a^\mu_n$$

implies that the number of physical states w/$\alpha' M^2$= (N-1) is given by the number of solutions of the equation (in the integers $N_{n,\mu}$):

$$\alpha' M^2 + 1 = N = \sum_{n,\mu=1}^{D} n N_{n,\mu}$$

- This is the famous "Partitio Numerorum" problem solved long ago by the Hardy-Ramanujan formula (for D =1). A much higher degeneracy than what one was expecting:

$$d(N) = N^{-p} e^{2\pi\sqrt{\frac{ND}{6}}} = N^{-p} e^{2\pi\sqrt{\frac{\alpha' D}{6}}M}$$

- Although unexpected, this was just the behaviour postulated by R. Hagedorn a few years earlier (~1965) on a more phenomenological basis (e.g. a Boltzmann factor in the "transverse energy" of particles produced in high energy hadronic collisions). Actually, the true degeneracy will be (after ghosts are eliminated):

$$d(N) = N^{-p} e^{2\pi\sqrt{\frac{N(D-2)}{6}}} = N^{-p} e^{2\pi\sqrt{\frac{\alpha'(D-2)}{6}}M}$$

- Furthermore the value of D will be fixed by consistency

- Taken at face value, such a density of states leads to a limiting (maximal, Hagedorn) temperature (FV, 1969) since:

$$Z(\beta) \equiv Tr[e^{-\beta H}] \sim \int dM \; d(M) e^{-\beta M} \sim \int dM \; e^{cM - \beta M}$$

diverges for $\beta = 1/(k_B T) < c$ giving the limiting temperature

$$T_H = \frac{1}{2\pi\sqrt{\alpha'}} \sqrt{\frac{6}{D-2}}$$

i.e. a maximal temperature of order a few hundred MeV (if we set D=4 and take for α' the experimental value)! Unfortunately, consistency will prevent us from taking D =4.

- T_H has an interesting reinterpretation in QCD as a deconfining temperature (quarks no longer bound inside hadrons)
- In string theory such an interpretation, so far, is absent.