# Cattedra Enrico Fermi 2015-2016

## La teoria delle stringhe:
## l'ultima rivoluzione in fisica?

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**Lezione # 5.1: 10.12.2015**

## L'oscillatore armonico classico e quantistico
## Il modello a risonanze duali

# Piano delle due prossime lezioni

- Il modello a risonanze duali (DRM) e un primo sguardo al suo possibile spettro.
- Condizione di Virasoro per l'eliminazione dei ghosts e vincolo sulle dimensioni spaziali.
- Indicazioni a favore di una stringa sottostante
- La stringa classica di Nambu-Goto
- Problemi fenomenologici e sopravvento di QCD

# Punti principali dell'ultima lezione

- Abbiamo visto come il "bootstrap" basato sulla dualità di DHS possa essere risolto esattamente in un caso particolarmente simmetrico ($\pi \pi \to \pi \omega$) in termini della funzione Beta di Eulero.
- Abbiamo poi discusso alcune proprietà della soluzione e, in particolare, come la somma sulle risonanze in un canale (ad es. $s$) fornisca necessariamente anche i poli nel canale crossato (ad es. $t$), l'espressione più semplice e sintetica della dualità.

# Preambolo

- La formulazione più semplice e utile della teoria delle stringhe evita il formalismo della teoria dei campi e si basa invece su un approccio di "prima quantizzazione" simile a quello della meccanica quantistica non relativistica.
- Dato che la stringa è un insieme di oscillatori armonici è necessario richiamare la quantizzazione di un oscillatore armonico (con le mie scuse agli esperti).

# Non-relativistic harmonic oscillator:

## 1. The classical case

The Lagrangian: $$L = \frac{1}{2}(m\dot{x}^2 - Tx^2) \equiv \frac{1}{2}m(\dot{x}^2 - \omega^2 x^2)$$

The Hamiltonian: $$H = \frac{1}{2}\left(\frac{p^2}{m} + m\,\omega^2 x^2\right) \qquad p = m\,\dot{x}$$

The equation of motion: $$\ddot{x} + \omega^2 x = 0$$

The general solution: $$x(t) = A\cos(\omega t + \varphi)$$

The (constant) energy: $$H = E = \frac{1}{2}TA^2$$

NB: the spectrum is continuous and has no gap: A=0 => E = 0

# 2. Dirac-style quantization

x and p become non-commuting operators (indicated by a hat)

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2, \qquad [\hat{x}, \hat{p}] = i\hbar \qquad \hat{p} = -i\hbar\frac{\partial}{\partial x}.$$

introduce auxiliary destruction and creation operators

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$$

(better defined without Planck's constant?)

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$$

$$[a, a^\dagger] = 1, \qquad [N, a^\dagger] = a^\dagger, \qquad [N, a] = -a,$$

$$H = \frac{1}{2}(a^\dagger a + a a^\dagger) = a^\dagger a + \frac{1}{2}\hbar\omega \equiv N + \frac{1}{2}\hbar\omega$$

Finding the spectrum becomes (almost) trivial…

$$H = \frac{1}{2}(a^\dagger a + a a^\dagger) = a^\dagger a + \frac{1}{2}\hbar\omega \equiv N + \frac{1}{2}\hbar\omega$$

using Dirac's bracket notation for states and checking existence of a "ground state" |0> (next page) we find its energy:

$$a\,|0\rangle = 0. \qquad\qquad H\,|0\rangle = \frac{\hbar\omega}{2}\,|0\rangle$$

Excited states are also easily obtained via:

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle. \qquad\qquad H\,|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle\,,$$

Each level is non-degenerate.

Completeness of eigenstates:

$$1 = \sum_{n=0}^{\infty} |n\rangle\langle n|$$

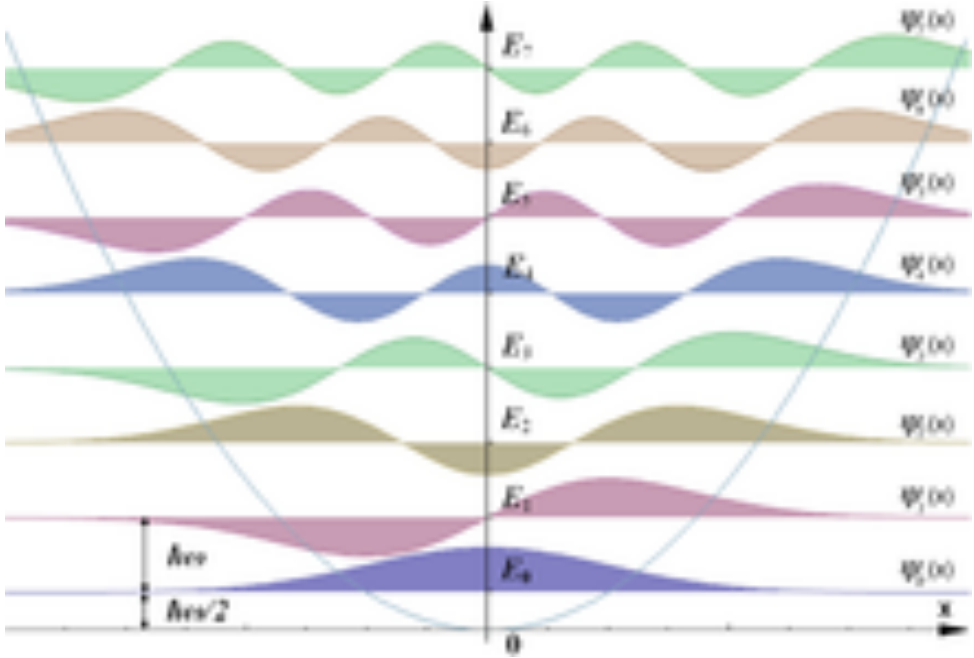
# Checking normalizability of ground state:

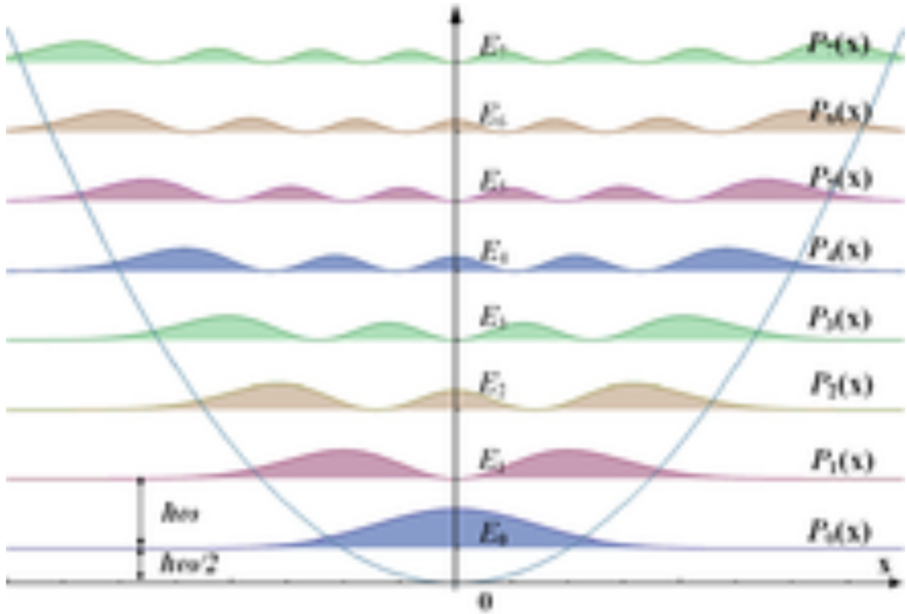
$$\langle x \mid a \mid 0 \rangle = 0 \qquad \Longrightarrow$$

$$\left( x + \frac{\hbar}{m\omega} \frac{d}{dx} \right) \langle x \mid 0 \rangle = 0 \qquad \Longrightarrow$$

$$\langle x \mid 0 \rangle = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \exp \left( -\frac{m\omega}{2\hbar} x^2 \right) = \psi_0 \ ,$$

# wave-functions and position probability of first 8 states



# The N-dimensional oscillator(dropping hats)

$$[x_i, p_j] = i\hbar\delta_{i,j}$$
$$[x_i, x_j] = 0$$
$$[p_i, p_j] = 0$$

The Hamiltonian:
$$H = \sum_{i=1}^{N} \left( \frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 x_i^2 \right).$$

$$a_i = \sqrt{\frac{m\omega}{2\hbar}} \left( x_i + \frac{i}{m\omega} p_i \right),$$
$$a_i^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left( x_i - \frac{i}{m\omega} p_i \right).$$

$$H = \hbar\omega \sum_{i=1}^{N} \left( a_i^\dagger a_i + \frac{1}{2} \right).$$

The spectrum:
$$E = \hbar\omega \left[ (n_1 + \cdots + n_N) + \frac{N}{2} \right].$$

Degeneracy of the $n^{th}$ excited level:
$$g_n = \binom{N+n-1}{n}$$

grows like $n^{(N-1)}$

# Due osservazioni

- A differenza del caso della teoria dei campi $a^+$ (a) non crea (distrugge) particelle ma solo eccitazioni (lo stesso sarà vero per la stringa)
- La relazione fra E e ω somiglia a quella fra frequenza ed energia di un fotone. In effetti un oscillatore armonico può effettuare transizioni fra un livello e un altro emettendo o assorbendo fotoni con l'energia corrispondente al salto di energia fra i due livelli:

$$\Delta E = \hbar(\Delta N)\omega$$

## Back to our scattering amplitudes!

# Fear of ghosts

- The properties of the Beta-function were very nice and welcome, almost too good to be true.
- There was, however, a big worry based on previous experience: possibly, in order to satisfy all the constraints, the model contained "ghosts", i.e. negative-norm states produced with negative probability.
- If so the model would have been inconsistent.
- In order to find out, it was necessary to identify first all the states.

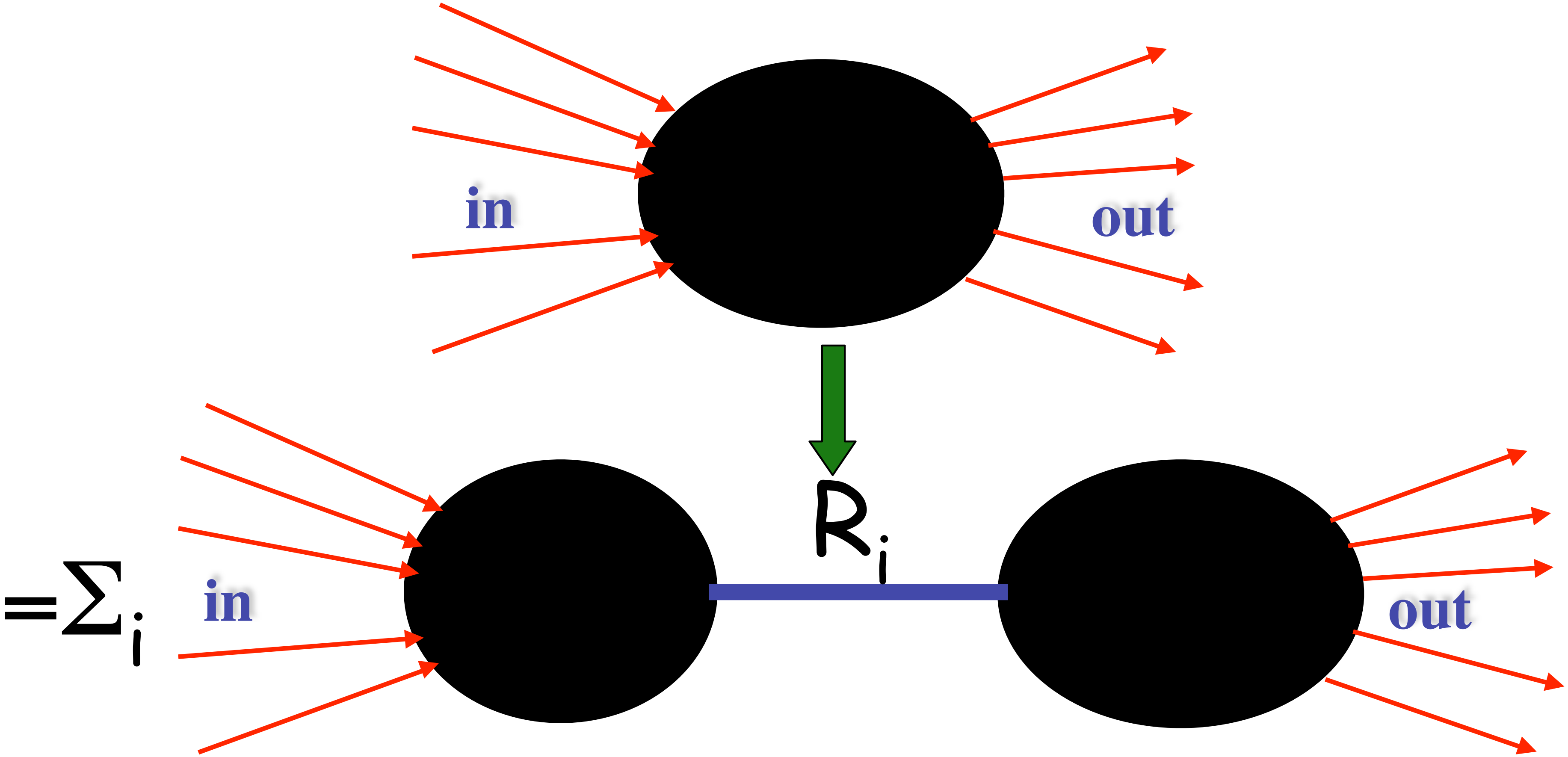
# A first look at the spectrum

Consider the $N^{th}$ pole in s, at $\alpha(s) = N$, and its residue.
It is immediately realized that such a residue is a polynomial in t (hence in cos θ) of degree N. As such it can be expanded in the first N Legendre polynomials each one corresponding to a definite J for the resonance.
One finds that all J up to N do indeed contribute. Thus the spectrum is degenerate with a degeneracy growing at least linearly (quadratically if we count 2J+1 states for spin J) in N.
At least, because a single 2->2 scattering process is unable to resolve the degeneracy within a given J.

To fully disentangle the spectrum we need to construct more general scattering amplitudes and use a basic property of each single intermediate state: factorization itself a consequence of unitarity. It says that each state contributes to the residue by the product of its couplings to the “initial” and “final” states.

=> Counting states amounts to answering the question:

Q: How many terms are needed (in the sum over i) in order to have, **for all in and out states,**



In principle we could stick to 2-->2 processes varying the species of the initial and final particles. In practice, the simplest extension turns out to be to increase the number of particles participating in the scattering process, without changing their nature.
This turns out to be good enough for our purposes.

# N-particle generalizations of the Beta function

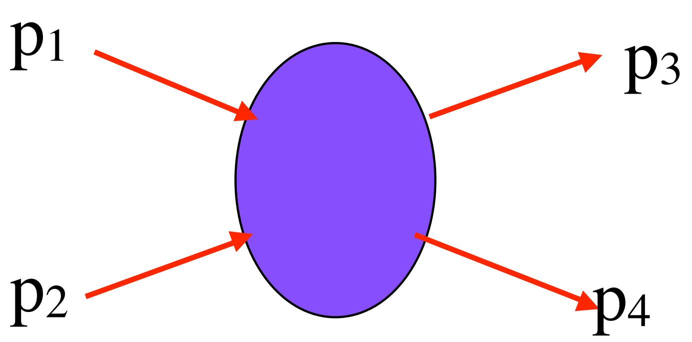
Which properties should our multiparticle amplitudes satisfy?
How should we generalize the duality properties of two-body scattering that allowed us to find the solution?
We will insist on having poles (and only poles) in the appropriate Mandelstam variables as well as the appropriate crossing symmetries, but will not impose Regge behaviour (which, btw, can be generalized to multiparticle processes).
Actually, (multi)Regge behaviour will come out as a bonus.
The other crucial input will be imposing "Planar Duality".

# Planar Duality (to be related to open strings)

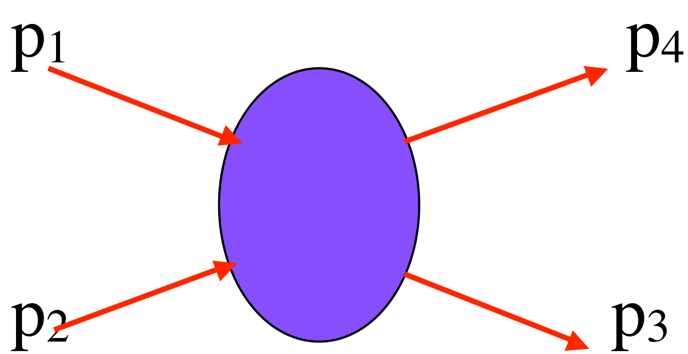
The Beta-function model for the 4-point function (2->2 scattering) exhibits "planar duality" i.e. duality w.r.t. the channels put in evidence by each particular cyclic order of the external lines. There are 3 of them (3 pairs of Mandelstam variables):

$$s = -(p_1 + p_2)^2 \quad , \quad t = -(p_1 - p_3)^2 \quad , \quad u = -(p_1 - p_4)^2$$

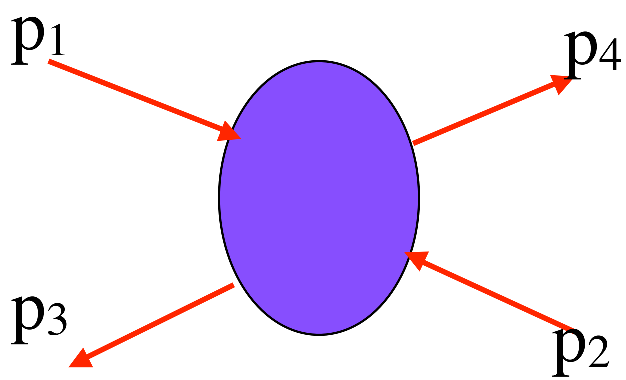
s-t duality



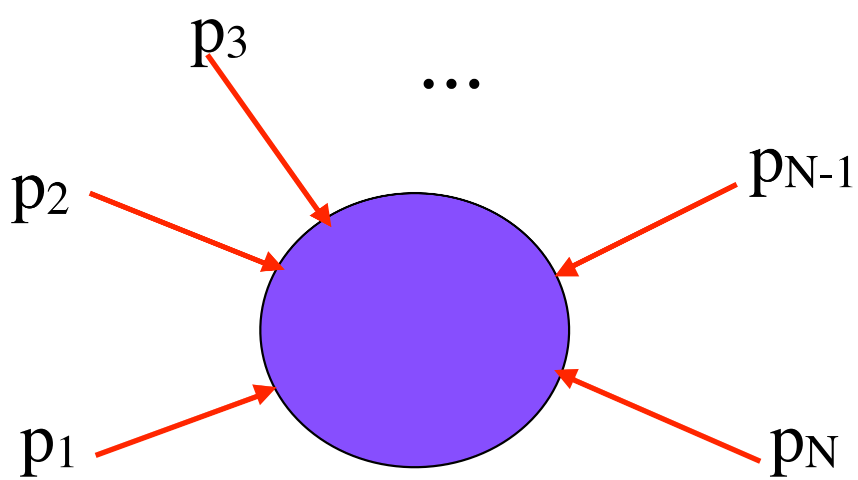

s-u duality




t-u duality



Consider now a process involving N > 4 external spinless particles. The corresponding (connected) amplitude is called an N-point function $A_N$. There are (N-1)!/2 distinct terms (distinct cyclic orderings) that have to be added at the end (with some specific numerical weights). Consider the term corresponding to the “natural” cyclic ordering:



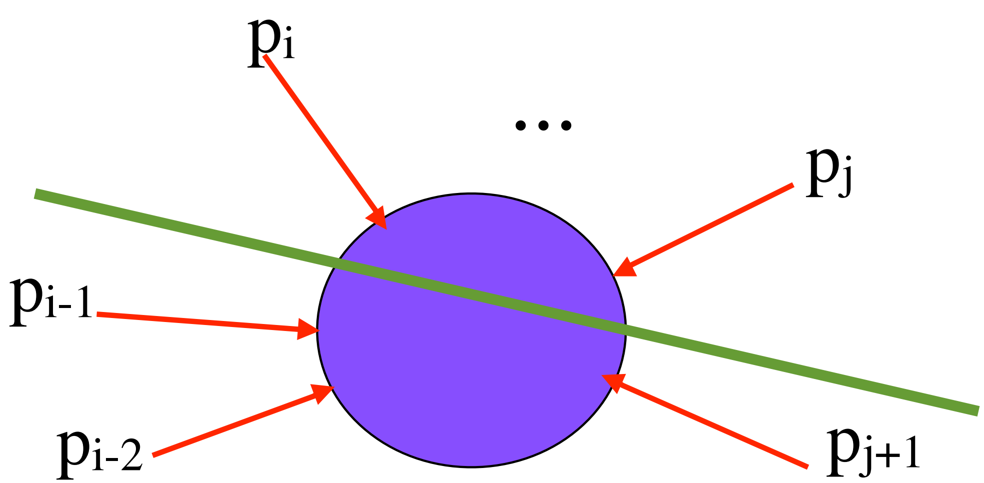

It will be given by an analytic function with poles in the Mandelstam invariants corresponding to its “planar channels”.

Useful convention: all momenta are incoming so that 4-momentum conservation reads: $\Sigma_i \ p_i = 0$.

=> some of the $p_{i0} = E_i$ must be negative.

They correspond to outgoing particles w/ 4-momentum $-p_i$

planar channels are defined by a partition of the external legs in two sets of adjacent legs each containing at least two particles

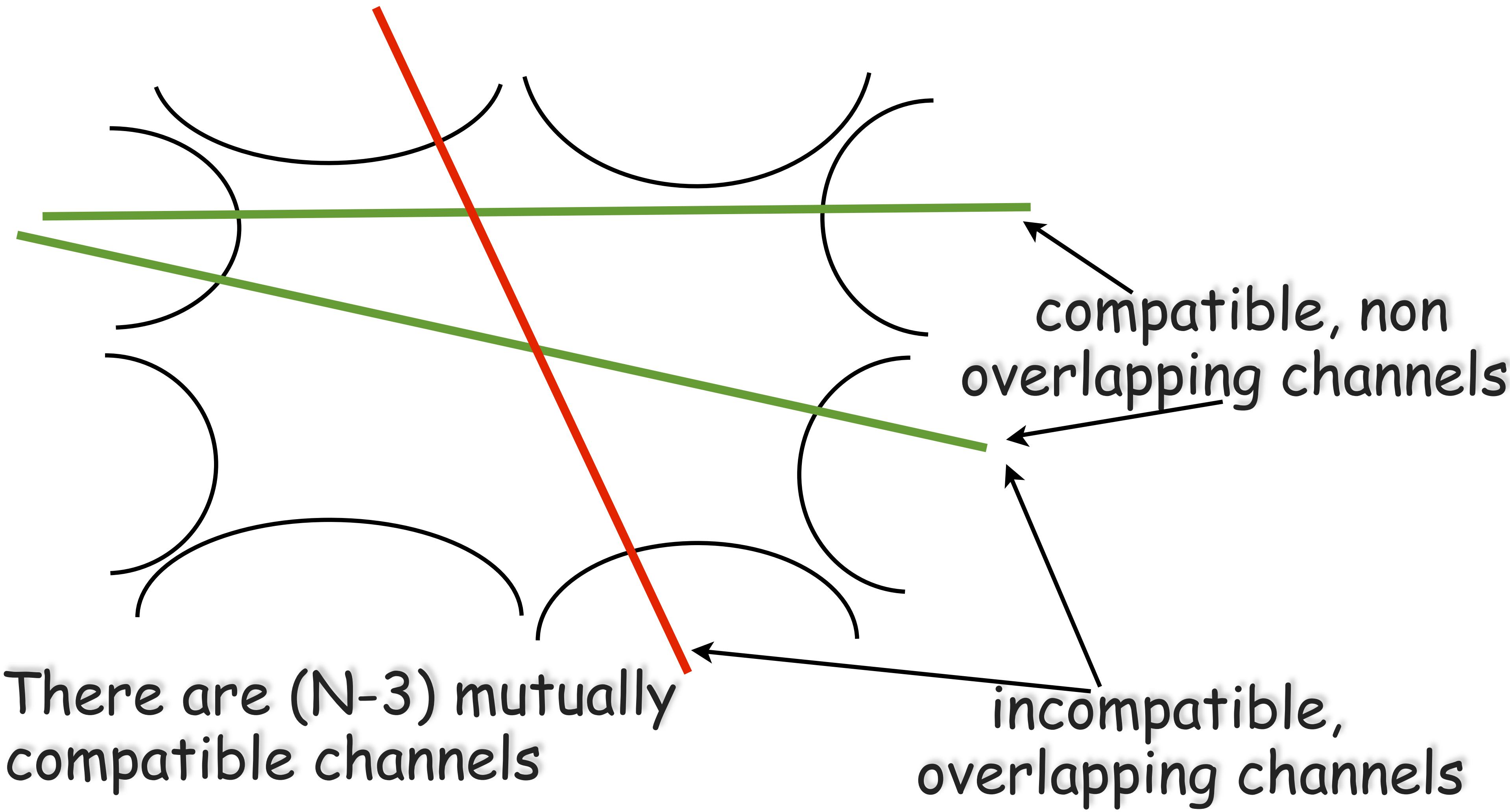



Poles appear in the corresponding Mandelstam variables:

$s_{ij} = -(p_i + p_{i+1} + \ldots + p_j)^2 = -(p_{j+1} + p_{j+2} + \ldots + p_{i-1})^2$

Their total number is $N(N-3)/2$ (= 2, 5, 9, ...)

# Planar duality is very natural from a duality diagram viewpoint



# Planar duality for N-particle amplitude

The sum over the poles in the Mandelstam variables of a maximal set of (N-3) mutually compatible channels should give the full scattering amplitude (including the poles in the overlapping channels)

Pausa ?