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La teoria delle stringhe: l'ultima rivoluzione in fisica?

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La stringa come relatività generale in 2 D
Stringhe classiche e quantistiche
Complementi tecnici

Complementi tecnici

1. Equivalenza delle due formulazioni
2. I vincoli nella formulazione di Polyakov.
3. Algebra classica dei vincoli
4. Aggiunta di cariche nella formulazione di Polyakov
5. Quantizzazione:
   - Quantizzazione sul cono luce
   - Quantizzazione alla Polyakov (solo note)
   - Quantizzazione alla BRST (solo note)

Equivalenza classica: 1. Point particle

There is a nice way to deal with both the massive and the massless case. It uses an extra field playing the role of the metric (actually: $e^2 = -g_{00}$) of a 1-dimensional GR.

$$\tilde{S}_p = \frac{1}{2}\int d\tau \left(e^{-1}\dot{x}^\mu \dot{x}^\nu g_{\mu\nu}(x) - em^2\right)$$

Using the eom for e:

$$e^2 = -\frac{1}{m^2}\dot{x}^\mu \dot{x}^\nu g_{\mu\nu}(x) \Rightarrow p_\mu p_\nu g^{\mu\nu}(x) = -m^2$$

we can eliminate it and show the classical equivalence of the two formulations.

It is also easy to prove that .

$$p_\mu p_\nu g^{\mu\nu} = -m^2$$

2. Polyakov's formulation of the bosonic string

$$S_P = -\frac{T}{2}\int d^2\xi \sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X)$$

After expressing $\gamma_{\alpha\beta}$ in terms of X^μ and $G_{\mu\nu}$ using its eom:

$$\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X) - \frac{1}{2}\gamma_{\alpha\beta}\left[\gamma^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X^\nu G_{\mu\nu}(X)\right] = 0$$

one gets back the NG action (where $\gamma_{\alpha\beta}$ was NOT an independent field):

$$S_{NG} = -T\int d^2\xi \sqrt{-det\gamma_{\alpha\beta}}$$

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu(\xi)}{\partial\xi^\alpha}\frac{\partial X^\nu(\xi)}{\partial\xi^\beta}G_{\mu\nu}(X(\xi))$$

This is what we get up to a conformal factor that drops out

Recovering the constraints in Polyakov's formulation

Absorbing a factor T in the definition of G:

$$S_P = -\frac{1}{2}\int d^2\xi\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu}(X)$$

the constraints following from the eom of $\gamma_{\alpha\beta}$:

$$\gamma_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}$$

can be easily written w/out choosing any gauge. Defining as usual the canonical momentum conjugate to X:

$$P_\mu(\sigma) = \frac{\delta S_P}{\delta \dot{X}^\mu} = -\sqrt{-\gamma}\gamma^{0\beta}\partial_\beta X^\nu G_{\mu\nu}(X)$$

one can easily prove the constraints:

$$P_\mu X'^\mu = 0 \;\; ; \;\; P_\mu P_\nu G^{\mu\nu}(X) + X'^\mu X'^\nu G_{\mu\nu}(X) = 0$$

For the momentum constraint the check is quite trivial:

$$P_\mu X'^\mu = -\sqrt{-\gamma}\gamma^{0\beta}\partial_\beta X^\nu G_{\mu\nu}(X)\partial_1 X^\mu \sim \gamma^{0\beta}\gamma_{\beta 1} = 0$$

For the Hamiltonian constraint the check is a little less trivial. Note that only this latter constraint contains explicitly the space-time metric.

As Dirac taught us long ago we should check the algebra of the constraints (via classical Poisson brackets). It is useful to take the sum and the difference of the constraints: $P_\mu X'^\mu = 0 \;\; ; \;\; P_\mu P_\nu G^{\mu\nu}(X) + X'^\mu X'^\nu G_{\mu\nu}(X) = 0$

$$4L_\pm = (P_\mu \pm G_{\mu\rho}X'^\rho)\; G^{\mu\nu}\; (P_\nu \pm G_{\nu\sigma}X'^\sigma) = 0$$

One then finds:

$$[L_\pm(\sigma), L_\pm(\sigma')]_{P.B.} = \pm(L_\pm(\sigma) + L_\pm(\sigma'))\,\delta'(\sigma - \sigma')$$

Mixed P.B. vanish. $\quad [X^\mu(\sigma,\tau), P_\nu(\sigma',\tau)]_{P.B.} = \delta^\mu_\nu \delta(\sigma - \sigma')$

Aggiunta di cariche

Aggiungendo l'interazione con un campo elettromagnetico:

$$\tilde{S}_p = \frac{1}{2}\int d\tau\,\left(e^{-1}\partial_\tau x^\mu \partial_\tau x^\nu g_{\mu\nu}(x) - e\, m^2\right) - q\int d\tau \frac{dx^\mu(\tau)}{d\tau} A_\mu(x)$$

si vede come quest'ultimo non necessita l'introduzione di "e". Analogamente si può aggiungere a S_P l'interazione con un tensore anti-simmetrico $B_{\mu\nu}$ e si ottiene (avendo assorbito qualche fattore in G):

$$S_P = \int d^2\xi \partial_\alpha X^\mu \partial_\beta X^\nu \left[\sqrt{-\gamma}\gamma^{\alpha\beta} G_{\mu\nu}(X) + \epsilon^{\alpha\beta} B_{\mu\nu}(X)\right]$$

si vede come quest'ultimo non necessiti l'introduzione di $\gamma_{\alpha\beta}$. Notare altrimenti come G e B entrino in modo simile nell'azione.

Quantization (open string)

In order to quantize the string we proceed as in any QFT and promote X and P to non-commuting operators:

$$X_\mu(\sigma,\tau) = q_\mu + 2\alpha' p_\mu \tau + i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left[\frac{a_{n,\mu}}{\sqrt{n}}e^{-in\tau} - \frac{a^\dagger_{n,\mu}}{\sqrt{n}}e^{in\tau}\right] cos(n\sigma)$$

$$\begin{aligned} X_\mu(\sigma,\tau) &= q_\mu + 2\alpha' p_\mu \tau + \frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left[\frac{a_{n,\mu}}{\sqrt{n}}e^{-2in(\tau-\sigma)} - \frac{a^\dagger_{n,\mu}}{\sqrt{n}}e^{2in(\tau-\sigma)}\right] \\ &+ \frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left[\frac{\tilde{a}_{n,\mu}}{\sqrt{n}}e^{-2in(\tau+\sigma)} - \frac{\tilde{a}^\dagger_{n,\mu}}{\sqrt{n}}e^{2in(\tau+\sigma)}\right] \end{aligned}$$

Using the standard h. osc. c.r. we get the desired result:

$$[X_\mu(\sigma,\tau), P_\nu(\sigma',\tau)] = i\eta_{\mu\nu}\delta(\sigma-\sigma') \quad , \quad (\hbar = 1)$$

The only tricky things to take care of are the constraints!

1. Light-cone quantization

(Goddard, Goldstone, Rebbi & Thorn, 1972)

The residual freedom to perform conformal transformations:

$$\tau \pm \sigma \longrightarrow f_\pm(\tau \pm \sigma)$$

allows us to (almost completely) fix one of the coordinates. For the rigid rod we took $X_0 = A\tau$, but for the general case it is more useful to fix instead:

$$X^+(\sigma,\tau) = 2\alpha' p^+ \tau \quad ; \quad X^\pm(\sigma,\tau) \equiv \frac{X^0 \pm X^{D-1}}{\sqrt{2}}$$

$$P^+(\sigma,\tau) \quad = \quad T\dot{X}^+(\sigma,\tau) = \frac{1}{\pi} p^+$$

The constraints can be solved for X^- since one must have:

$$2\dot{X}^- \dot{X}^+ = \sum_{i=1}^{(D-2)} \left(\dot{X}_i^2 + {X_i'}^2 \right) \; ; \; X'^- \dot{X}^+ = \sum_{i=1}^{(D-2)} \dot{X}_i X_i'$$

$$\dot{X}^- = \frac{1}{4\alpha' p^+} \sum_{i=1}^{(D-2)} \left(\dot{X}_i^2 + X_i'^2 \right) \; ; \; X'^- = \frac{1}{2\alpha' p^+} \sum_{i=1}^{(D-2)} \dot{X}_i X_i'$$

These equations allow to express the a^-_n oscillators in terms of the transverse ones (while the a^+_n are zero). Note that the a^-_n oscillators become bilinear in the transverse ones.

Therefore in this gauge we were able to solve the constraints and to reduce the physical spectrum to the one generated by (D-2) space-like oscillators.

At this point it looks as if we managed to eliminate all the ghosts without getting any constraint on a_0 or on D.

The problem is that the l.c. gauge breaks explicit Lorentz invariance: we have to check that L.I. is still there, even if hidden ...

A Lorentz anomaly?

We have to check the O(D-1,1) Lorentz algebra:

$$i[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho} M_{\nu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma}$$

$$M_{\mu\nu} \equiv \int d\sigma \, (P_\mu X_\nu - P_\nu X_\mu)$$

The check is easy for the compact O(D-2) subgroup (L_{ij} are bilinear in the oscillators) but becomes non-trivial for the components of the Lorentz generators involving the ± directions (these may involve three oscillators). In particular $[M_{+i}, M_{+j}]$ should vanish (recall that $\eta_{++} = 0$) while a long but straightforward calculation by GGRT gives:

$$[M_{+i}, M_{+j}] \propto \sum_{n=1}^{\infty} \left[n^2 \left(\frac{D-2}{24} - 1 \right) - \left(\frac{D-2}{24} - \alpha_0 \right) \right] \left(a_n^{\dagger i} a_n^j - a_n^{\dagger j} a_n^i \right)$$

We thus find that the Lorentz algebra is broken unless $\alpha_0 = 1$ and $D = 26$, i.e. the same conditions we found in the DRM!

2. Polyakov's approach

P. started from the formal path integral:

$$``Z" = \int [dX d\gamma] \exp\left(-S_E(X, \gamma_{\alpha\beta})\right)$$

Since the action is invariant under 2D diffs and Weyl this integral if actually infinite. We have to fix the gauge and define the functional integral over a gauge-slice. In order to get a result independent of the slice/gauge, we have to add a Fadeev-Popov determinant, i.e. the Jacobian for going from $[d^3\gamma]$ to $[d^3\zeta]$ where ζ^i are 3 gauge-transf. parameters:

But do we really get a result which does not depend on the gauge? We have to compute carefully the integral. This is done by introducing anticommuting fields (called FP ghosts but with nothing to do with our previous ghosts), since integrals over (bosons)fermions give (1/det) det of the operators appearing in their quadratic actions.

The result of this rather non-trivial calculation is the following: one can regularize the theory in such a way that 2d-diffs are preserved. That means that Z will depend, at most, on the conformal factor appearing in:

$$\gamma_{\alpha\beta} \to e^{2\lambda(\sigma,\tau)}\gamma_{\alpha\beta}$$

A correct derivation should be as follows:

$$\text{“}Z\text{”} = \int [dX d\gamma] \exp\left(-S_E(X, \gamma_{\alpha\beta})\right) \qquad \text{replaced by}$$

$$Z = \int \frac{[dX d\gamma]}{V_{diff.\times Weyl}} \exp\left(-S_E(X, \gamma_{\alpha\beta})\right)$$

Defining $\Delta_{FP}(\gamma, \hat{\gamma})^{-1} \equiv \int [d\zeta]\delta(\zeta\gamma - \hat{\gamma}) = |det \frac{\partial \zeta\gamma}{\partial \zeta}|^{-1}_{\zeta\gamma = \hat{\gamma}}$

By the invariance of the group measure

$$\Delta_{FP}(\zeta\gamma, \hat{\gamma})^{-1} = \Delta_{FP}(\gamma, \hat{\gamma})^{-1}$$

and we obtain

$$Z = \int \frac{[dX d\gamma]}{V_{diff.\times Weyl}} \exp\left(-S_E(X, \gamma_{\alpha\beta})\right) = \int [dX]\Delta_{FP}(\hat{\gamma}, \hat{\gamma}) \exp\left(-S_E(X, \hat{\gamma})\right)$$

Finally:

$$\Delta_{FP}(\hat{\gamma}, \hat{\gamma}) = |det \frac{\partial \zeta\gamma}{\partial \zeta}|_{\zeta\gamma = \hat{\gamma}} = \int [db_i dc^i] \exp\left[b_i \left(\frac{\partial \zeta\gamma}{\partial \zeta}(\zeta\gamma = \hat{\gamma}) \right)_{ij} c^j \right]$$

Indeed one finds (Polyakov 1981)

$$Z(\lambda) = Z(0) \exp\left(\frac{c}{24\pi}\int d^2\xi \partial_\alpha \lambda \partial^\alpha \lambda\right) \quad ; \quad \text{with } c = D - 26.$$

This can be written in a covariant (but non-local) form:

$$Z(\gamma) \sim \exp\left(-\frac{c}{96\pi}\int d^2\xi \gamma^{1/2} R \,\Box^{-1} R\right)$$

since for a conformally flat 2d-metric:

$$\gamma^{1/2} R = -2\,\Box_f \lambda \;\; ; \;\; \Box = e^{-2\lambda}\Box_f$$

The non-invariance of the action wrt Weyl transformations can also be translated into a statement about the non-vanishing of the trace of $T_{\alpha\beta}$. One finds:

$$T_\alpha^\alpha = -\frac{c}{12} R$$

In order to get also the constraint on α_0 one needs to consider interactions...

3. The BRST Hamiltonian approach

Starting from the constraints and their algebra:

$$P_\mu X'^\mu = 0 \quad ; \quad P_\mu P_\nu G^{\mu\nu}(X) + X'^\mu X'^\nu G_{\mu\nu}(X) = 0$$

$$4L_\pm = (P_\mu \pm G_{\mu\rho} X'^\rho)\, G^{\mu\nu}\, (P_\nu \pm G_{\nu\sigma} X'^\sigma) = 0$$

$$[L_\pm(\sigma), L_\pm(\sigma')]_{P.B.} = \pm (L_\pm(\sigma) + L_\pm(\sigma'))\, \delta'(\sigma - \sigma')$$

Note that:

a. The algebra does not depend on the background metric

b. The algebra closes on the constraints themselves with constant coefficients.

c. The canonical Hamiltonian is zero.

a. and b. allow to quantize the system following a simplified Batalin-Fradkin-Vilkovisky approach.
For a finite set of (bosonic) constraints G_i with algebra:

$$[G_i, G_j] = i g_{ij}^k G_k$$

1. Introduce a pair of conjugate Grassmann variables (c^i, b_i) for each G_i with $\{c^i, b_j\} = \delta^i_j$

2. Construct a (classically nilpotent) BRST operator Q:

$$Q_{BRST} = Q = c^i G_i - \frac{i}{2} g_{ij}^k c^i c^j b_k = Q^\dagger \qquad \{Q, Q\}_{P.B.} = 0$$

3 Choose a gauge-fixing fermion χ and obtain the "total" Hamiltonian as:

$$H_{tot.} = H_{can.} + h^i_j b_i c^j + \{\chi, Q\} \; ; \; [H_{can}, G_i]_{P.B.} = h^j_i G_j$$

It commutes with Q.

4. Quantize canonically insisting on finiteness and check that the classical properties (in particular $Q^2 = 0$) still hold. If so you win (if not you have a gauge anomaly...) and:

i. Physical operators are those commuting with Q,

ii. Physical states are annihilated by Q.

iii Q acting on any state gives a "decoupled" physical state

iv Matrix elements between physical states do not depend on the choice of χ

In the case of the bosonic string, generalizing the procedure to an infinite set of constraints, we find:

$$Q = \int d\sigma \left(L_+ c^+ + L_- c^- + b_+ (c^+)' c^+ - b_- (c^-)' c^- \right)$$

For a non-trivial metric the check of Q^2=0 is very non-trivial but for a flat space-time it's relatively easy.

Consider just the + sector. Anomalies come from "double contractions" e.g.

$$(P + X')^2 c^+(\sigma) \times (P + X')^2 c^+(\sigma') \sim \frac{D}{(z - z')^4} c^+(\sigma) c^+(\sigma') \; ; \; z = e^{i\sigma}$$

Mixed products do not give anomalous contributions, but squares of b-c terms do. Schematically:

$$b_+ (c^+)' c^+(\sigma) \times b_+ (c^+)' c^+(\sigma') \sim -\frac{1}{(z - z')^4} c^+(\sigma) c^+(\sigma') +$$

$$+ \quad \frac{1}{(z - z')^3} [c^+(\sigma)(c^+)'(\sigma') - (c^+)'(\sigma) c^+(\sigma')] + \frac{1}{(z - z')^2} (c^+)'(\sigma)(c^+)'(\sigma')$$

Integrating by parts (and keeping careful track of signs) one can see a “13” emerging...

Insisting that all terms cancel (even those with milder small-distance behavior) gives back both D = 26 and α_0 =1