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La teoria delle stringhe: l'ultima rivoluzione in fisica?

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Punti materiali e stringhe classiche

Complementi: aggiunta di "cariche", vincoli, e.d.m. in generale, condizioni ai bordi, …

Particella puntiforme relativistica con massa

The action for a (massive) relativistic particle can be written in a general (but given, fixed) spacetime metric $g_{\mu\nu}(x)$ and for any D as:

$$S_p = -mc \int ds = -mc \int d\tau \sqrt{-\frac{dx^\mu(\tau)}{d\tau}\frac{dx^\nu(\tau)}{d\tau} g_{\mu\nu}(x(\tau))}$$

The action of a point-particle is thus proportional (with mc as the proportionality constant) to the proper length of the "world-line" described by the particle's motion and parametrized by $x^\mu(\tau)$. The classical motion is the one minimizing that length (a geodesic in the given metric).

Different particles can be introduced by adding similar terms each one with its own mass.

Interactions among particles are highly non unique and not always very easy to introduce, particularly if we want them among particles and local.

Instead, interactions with an electromagnetic field is easy to add via a term:

$$S_p = -mc \int ds - q \int d\tau \frac{dx^\mu(\tau)}{d\tau} A_\mu(x) = -mc \int ds - q \int dx^\mu A_\mu(x)$$

NB: un punto materiale si accoppia in modo “naturale” (senza invocare la metrica) a un campo vettoriale, una 1-forma.

La stringa bosonica nella formulazione di Nambu-Goto

The Nambu-Goto action for the relativistic string is a straightforward generalization of the previous action

Nambu (and independently Goto) wrote a geometric action proportional to the area of the surface ("world-sheet" in analogy with "world-line") swept by the string. T, the string tension, is the proportionality constant. Has dimensions of energy/length.

The string motion is parametrized by $X^{\mu}(\sigma, \tau)$ where:

$\mu = 0, 1, \ldots D-1$; $0 < \sigma < \pi$ (by convention), τ unconstrained.

NG did this in Minkowski spacetime ($g_{\mu\nu}(x) = \eta_{\mu\nu}(x)$) but, like for point particles, the construction can be easily generalized to an arbitrary metric $G_{\mu\nu}(x)$ and to any D.

$$S_{NG} = -T \int d(Area) = -T \int d^2\xi \sqrt{-det\gamma_{\alpha\beta}} \equiv -T \int d\xi^0 \int_0^\pi d\xi^1 \sqrt{-det\gamma_{\alpha\beta}}$$

where $\gamma_{\alpha\beta}$ is the "induced metric"

$$\gamma_{\alpha\beta} \equiv \frac{\partial X^\mu(\xi)}{\partial \xi^\alpha} \frac{\partial X^\nu(\xi)}{\partial \xi^\beta} G_{\mu\nu}(X(\xi)) \ , \quad \alpha, \beta = 0, 1 \quad , \quad \xi^0 = \tau, \ \xi^1 = \sigma$$

Again by analogy with the point-particle, the classical motion of the string is obtained by varying the action and corresponds to minimizing the area of the surface swept.

But, unlike in the case of point-particles, the problem is already non trivial even in Minkowski space-time (at the quantum level).

Another major difference: for point particles we can add in the action different particles of different mass thus introducing many free parameters.

Also, interactions among particles have to be added by hand (and it is not so simple!) and are quite arbitrary.

This is not the case for the string: there is just one T and interactions are automatically included in a "geometric" way!

What's the analog of the electromagnetic coupling of a particle for a string?

$$S_{string} = -T \int d^2\xi \sqrt{-det\gamma_{\alpha\beta}} + \int d^2\xi \epsilon^{\alpha\beta} \frac{\partial X^\mu(\xi)}{\partial \xi^\alpha} \frac{\partial X^\nu(\xi)}{\partial \xi^\beta} B_{\mu\nu}(X)$$

$$\gamma_{\alpha\beta} \equiv \frac{\partial X^\mu(\xi)}{\partial \xi^\alpha} \frac{\partial X^\nu(\xi)}{\partial \xi^\beta} G_{\mu\nu}(X(\xi)) \ , \quad \alpha, \beta = 0, 1 \quad , \quad \xi^0 = \tau, \ \xi^1 = \sigma$$

where $\varepsilon_{\alpha\beta}$ is the 2-d Levi-Civita tensor and $B_{\mu\nu} = - B_{\nu\mu}$ is an antisymmetric tensor field, a 2-form. Strings are naturally "charged" wrt such a field.

Note again the absence of ugly square roots…

The classical constraints

S_p is invariant under reparametrization of the world-line, $\tau \to \tau'(\tau)$. This leads to 1 constraint (easy to check):

$$p_\mu(\tau) \equiv \frac{\delta S_p}{\delta \dot{x}^\mu(\tau)} \Rightarrow p_\mu(\tau) p_\nu(\tau) g^{\mu\nu}(x(\tau)) = -m^2$$

Similarly, S_{NG} is invariant under reparametrization of the world-sheet by an arbitrary redefinition $\xi^\alpha \to \xi'^\alpha(\xi^\alpha)$

This leads now to 2 constraints (easy again to check):

$$P_\mu(\xi) \equiv \frac{\delta S_{NG}}{\delta \dot{X}^\mu(\xi)} \Rightarrow P_\mu(\xi) X'^\mu(\xi) = 0$$

$$P_\mu(\xi) P_\nu(\xi) G^{\mu\nu}(X(\xi)) + T^2 X'^\mu(\xi) X'^\nu(\xi) G_{\mu\nu}(X(\xi)) = 0$$

$$\dot{X}^\mu(\xi) \equiv \frac{\partial X^\mu(\xi)}{\partial \xi^0}\,, \quad X'^\mu(\xi) \equiv \frac{\partial X^\mu(\xi)}{\partial \xi^1}$$

Strings in Minkowski spacetime: action and equations of motion

Since classical string motion (and even more quantization) is already non-trivial in Minkowski spacetime let us consider that case (also needed for connection with DRM) but let's keep the dimensionality of spacetime D arbitrary.

$$S_{NG} = -T \int d^2\xi \sqrt{-det\gamma_{\alpha\beta}}$$

$$\gamma_{\alpha\beta} \equiv \frac{\partial X^\mu(\xi)}{\partial \xi^\alpha} \frac{\partial X^\nu(\xi)}{\partial \xi^\beta} G_{\mu\nu}(X(\xi)) \ , \quad \alpha, \beta = 0, 1 \quad , \quad \xi^0 = \tau, \ \xi^1 = \sigma$$

becomes:

$$S_{NG} = -T \int d^2\xi \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}$$

Lorentz product understood

$$\dot{X}^\mu(\xi) \equiv \frac{\partial X^\mu(\xi)}{\partial \tau} \ , \quad X'^\mu(\xi) \equiv \frac{\partial X^\mu(\xi)}{\partial \sigma}$$

The equations of motion for the point-particle are trivial while for the string they look quite frightening:

$$\delta S_{NG} \propto \int d\tau \int_0^\pi d\sigma \left[\left(\frac{\partial}{\partial \tau} \frac{\partial L}{\partial \dot{X}^\mu} + \frac{\partial}{\partial \sigma} \frac{\partial L}{\partial X'^\mu} \right) \delta X^\mu - \frac{\partial}{\partial \sigma} \left(\frac{\partial L}{\partial X'^\mu} \delta X^\mu \right) \right]$$

$$\begin{aligned} \frac{\partial}{\partial \tau} \frac{\partial L}{\partial \dot{X}^\mu} \quad &+ \quad \frac{\partial}{\partial \sigma} \frac{\partial L}{\partial X'^\mu} = 0 \\ \frac{\partial L}{\partial \dot{X}^\mu} \quad &= \quad T \frac{\dot{X}_\mu X'^2 - X'_\mu (\dot{X} \cdot X')}{\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}} \\ \frac{\partial L}{\partial X'^\mu} \quad &= \quad T \frac{X'_\mu \dot{X}^2 - \dot{X}_\mu (\dot{X} \cdot X')}{\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}} \end{aligned}$$

Boundary conditions

We need, at all τ,

$$\left(\frac{\partial L}{\partial X'^{\mu}}\delta X^{\mu}\right)(\sigma = 0) = \left(\frac{\partial L}{\partial X'^{\mu}}\delta X^{\mu}\right)(\sigma = \pi) \quad ; \quad (\text{no sum over } \mu)$$

For closed strings the points $\sigma = 0$ and $\sigma = \pi$ are physically the same point. If spacetime is topologically trivial this implies $X^{\mu}(0, \tau) = X^{\mu}(\pi, \tau)$ and the b.c. is satisfied.

For open strings we have two options (that can be used independently for each μ):

Neumann b.c. $\dfrac{\partial L}{\partial X'^{\mu}} = 0 \quad , \quad \sigma = 0, \pi$

Dirichlet b.c. $\delta X^{\mu} = 0 \quad , \quad \sigma = 0, \pi$

For the moment we will consider N. b.c. for open strings

A convenient choice of coordinates

There is a lot of freedom in the choice of the two WS coordinates (σ, τ). It can be used to impose two useful conditions (defining the orthonormal gauges):

$$\dot{X}^2 + X'^2 \equiv -(\dot{X}_0)^2 + (\dot{X}_i)^2 - (X'_0)^2 + (X'_i)^2 = 0$$
$$\dot{X} \cdot X' = 0 \quad , \quad \text{i.e.} \quad (\dot{\mathrm{X}} \pm \mathrm{X}')^2 = 0$$

The equations of motion become simply: $\ddot{X}_\mu = X''_\mu$

w/ solution: $X_\mu(\sigma, \tau) = F_\mu(\tau - \sigma) + G_\mu(\tau + \sigma)$

The boundary conditions for open strings also simply:

N b.c. $X'^\mu = 0 \quad , \quad \sigma = 0, \pi$

D b.c. $\dot{X}^\mu = 0 \quad , \quad \sigma = 0, \pi$

General solution of $\ddot{X}_\mu = X''_\mu$

Open (Neumann) strings ($X'_\mu(\sigma=0, \pi) = 0$). Def. $2\pi\alpha' = 1/T$

$$X_\mu(\sigma,\tau) = q_\mu + 2\alpha' p_\mu \tau + i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left[\frac{a_{n,\mu}}{\sqrt{n}}e^{-in\tau} - \frac{a^*_{n,\mu}}{\sqrt{n}}e^{in\tau}\right] cos(n\sigma)$$

Closed strings $X_\mu(\sigma=0) = X_\mu(\sigma=\pi)$

$$\begin{aligned} X_\mu(\sigma,\tau) &= q_\mu + 2\alpha' p_\mu \tau + \frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left[\frac{a_{n,\mu}}{\sqrt{n}}e^{-2in(\tau-\sigma)} - \frac{a^*_{n,\mu}}{\sqrt{n}}e^{2in(\tau-\sigma)}\right] \\ &+ \frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left[\frac{\tilde{a}_{n,\mu}}{\sqrt{n}}e^{-2in(\tau+\sigma)} - \frac{\tilde{a}^*_{n,\mu}}{\sqrt{n}}e^{2in(\tau+\sigma)}\right] \end{aligned}$$

to be added

$$\begin{aligned} \dot{X}\cdot X' &= 0 \\ \dot{X}^2 + X'^2 &\equiv -(\dot{X}_0)^2 + (\dot{X}_i)^2 - (X'_0)^2 + (X'_i)^2 = 0 \end{aligned}$$

Simplest classical solution: open string, the rotating rod

The equations to be solved are $\ddot{X}_\mu = X''_\mu$

subject to the constraints

$$\begin{aligned} \dot{X} \cdot X' &= 0 \\ \dot{X}^2 + X'^2 &\equiv -(\dot{X}_0)^2 + (\dot{X}_i)^2 - (X'_0)^2 + (X'_i)^2 = 0 \end{aligned}$$

and to the b.c. $X'_\mu(\sigma = 0, \pi) = 0 \Rightarrow \sum_i \left(\frac{dX_i}{dX_0}\right)^2 (\sigma = 0, \pi) = 1$

Ends move with speed of light. A simple solution is:

$$X_0 = A\tau \quad ; \quad X_1 = A\ cos\tau\ cos\sigma\ ,\ X_2 = A\ sin\tau\ cos\sigma$$

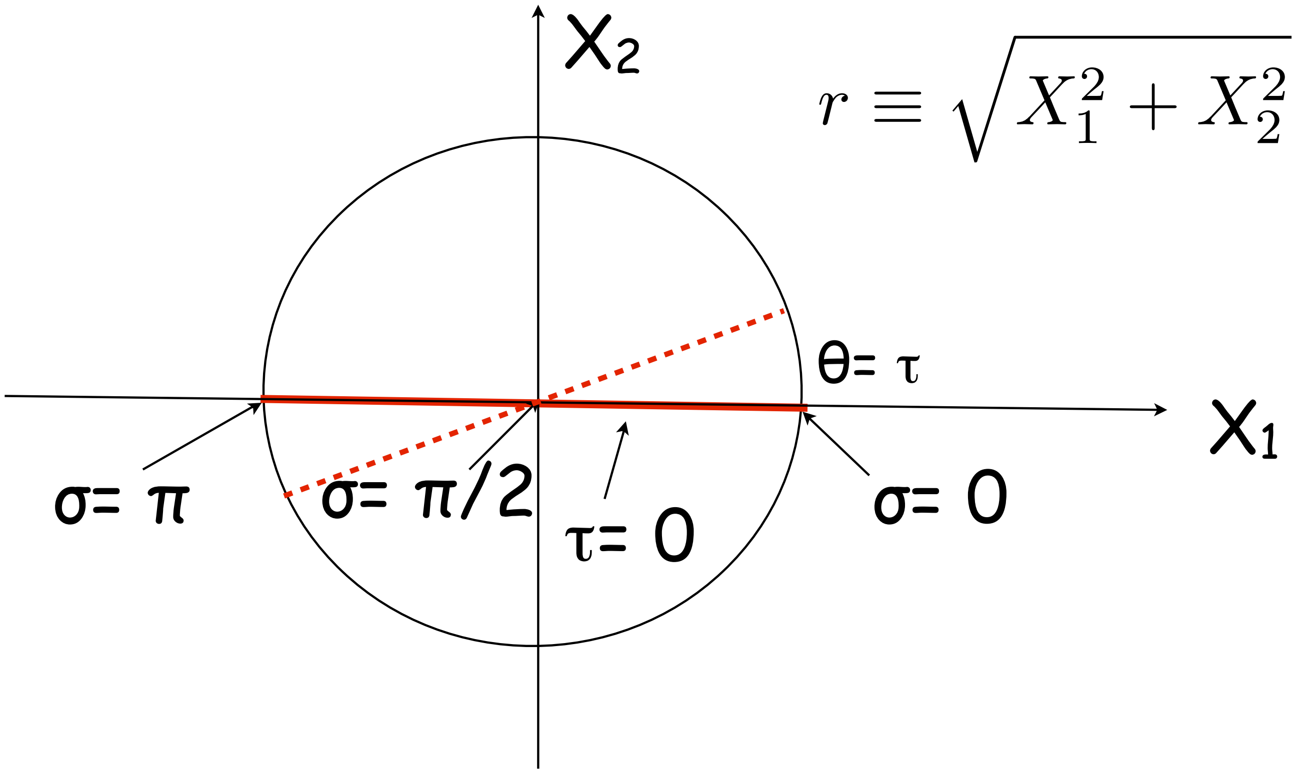
$$X_i = 0 \ , \ (i = 3, 4, \ldots D - 1)$$

e.o.m., constraints and b.c. easily checked!

$$X_0 = A\tau \quad ; \quad X_1 = A\ cos\tau\ cos\sigma\ ,\ X_2 = A\ sin\tau\ cos\sigma$$

$$X_i \ = \ 0\ ,\ (i = 3, 4, \ldots D-1)$$

$$r \equiv \sqrt{X_1^2 + X_2^2} = A|cos\sigma|$$



A rigid, rotating rod whose ends move with the speed of light since $dl/dX_0 = r\ d\theta/Ad\tau = r/A = |cos\sigma|$.

Let us now compute the energy (mass) and angular momentum of this classical string.

$$X_0 = A\tau \quad ; \quad X_1 = A\ cos\tau\ cos\sigma\ ,\ X_2 = A\ sin\tau\ cos\sigma$$

$$X_i \quad = \quad 0 \quad , \quad (i = 3, 4, \ldots D-1)$$

$$p_i = \int_0^\pi d\sigma P_i(\sigma) = T \int_0^\pi d\sigma \dot{X}_i = 0$$

$$E = M = \int_0^\pi d\sigma P_0(\sigma) = T \int_0^\pi d\sigma \dot{X}_0 = \pi T A$$

$$\begin{aligned} J_{12} \quad &= \quad \int_0^\pi d\sigma (X_1 P_2 - X_2 P_1)(\sigma) \\ &= \quad TA^2 \int_0^\pi d\sigma (sin^2\tau cos^2\sigma + cos^2\tau cos^2\sigma) = \frac{\pi}{2} TA^2 \end{aligned}$$

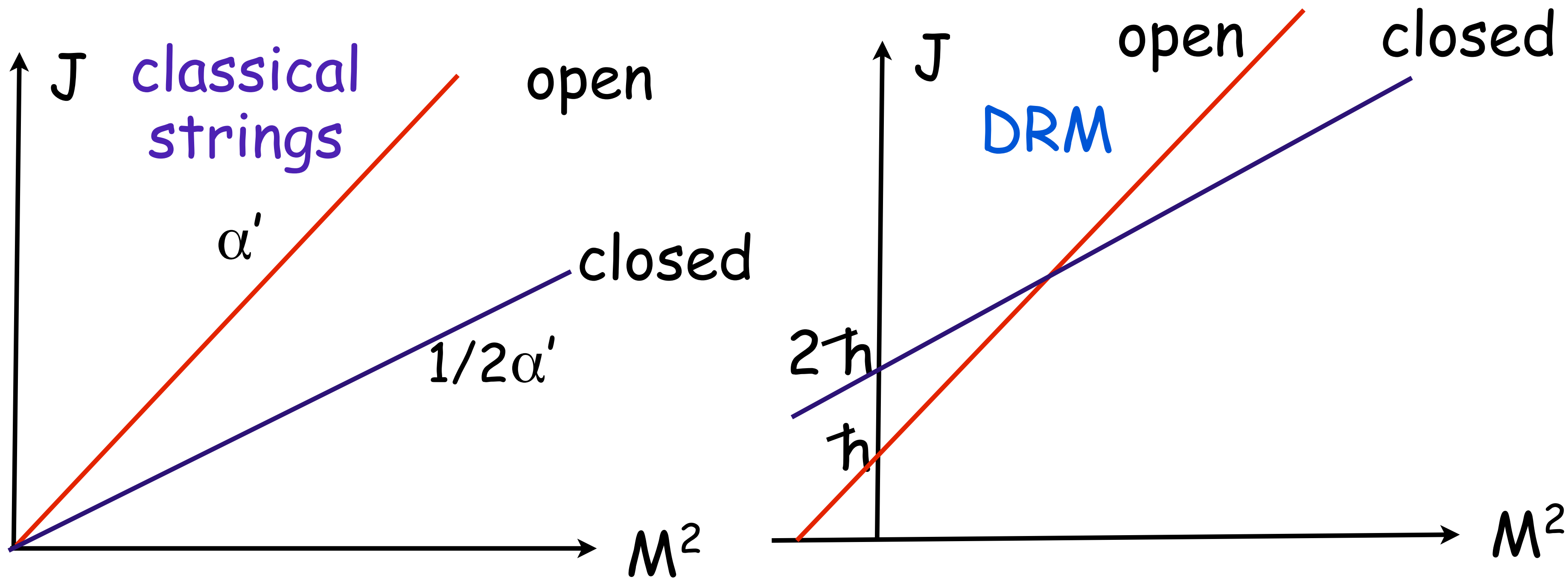
thus
$$J = \frac{M^2}{2\pi T} = \alpha' M^2\ ,\ \alpha' \equiv \frac{1}{2\pi T}$$

It is quite obvious that this solution maximizes the ratio J/M^2. The relation is very similar to the one given by the linear Regge trajectory we have been discussing in DRM.

For closed strings one finds the same relation between J and M^2 except for T --> 2 T (simple interpretation: for the same total length the closed string is half as big since it has to "come back on itself"). Hence α' -> 1/2 α'

Yet classical strings and DRM differ in crucial way.

In the classical theory, J and M^2 can take any real value with $J < \alpha' M^2$ => classical strings cannot have J without having mass! But in the DRM there were such states:



- The discrepancy between strings and the DRM disappears completely once we move from classical to quantum strings.
- This is where points and strings start to differ in a fundamental, qualitative way.
- Point particles can be quantized in an arbitrary background metric. This turns out not to be true for strings!
- As we shall see, even flat spacetime is in general forbidden... unless D takes a (so-called) critical value!
- String quantization is not trivial and can be done in many different ways. But the end result is always the same.