Spreading dynamics in graphs

PhD Student **Michele Sugarelli**

Supervisors Prof. **Angelo Vulpiani**
Dr. **Davide Vergni**

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Overview

- Reaction Diffusion problem
- Graph theory
- Reaction Diffusion dynamics on graphs
- Relation between topological structure of graphs and reaction dynamics
- Perspectives

Reaction Diffusion problem

Ubiquitous processes including reaction and diffusion that show up as soon as diffusing elements when propagating, get close and react, modifying their state

Epidemics spreading

Combustion

Ecology

Transport
in living organism

Oceanography

Two basic approaches:

- Single individuals moving and interacting (agent based description)

- Continuous fields diffusing and interacting in a media represented by mesoscopic and coarse-grained equation (pde description)

Reaction Diffusion problem

$$\partial_t \theta_\beta(\bar{x},t) = \sum_{i,j}\sum_{\gamma} D_{ij}^{\beta\gamma}\,\partial_{ij}^2\,\theta_\gamma(\bar{x},t) + \alpha_\beta F_\beta[\theta(\bar{x},t)]$$

FKPP model (1937)

$$\frac{\partial}{\partial t}\theta(x,t) = D\frac{\partial^2}{\partial x^2}\theta(x,t) + \alpha F[\theta(x,t)]$$

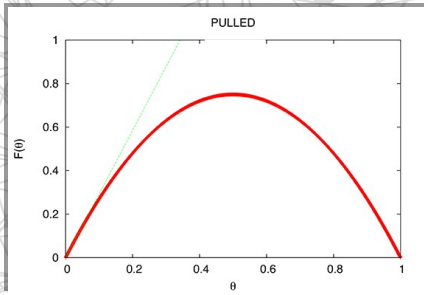
$$\theta(x,t)\in[0,1] \qquad F(0)=F(1)=0 \qquad F(\theta)>0 \;\; if \;\; 0<\theta<1$$

$$\begin{cases} \dfrac{d\theta_a}{dt} = \alpha\theta_a\theta_b \\[2ex] \dfrac{d\theta_b}{dt} = -\alpha\theta_a\theta_b \end{cases}$$

Autocatalytic reaction

$$A + B \rightarrow 2A$$

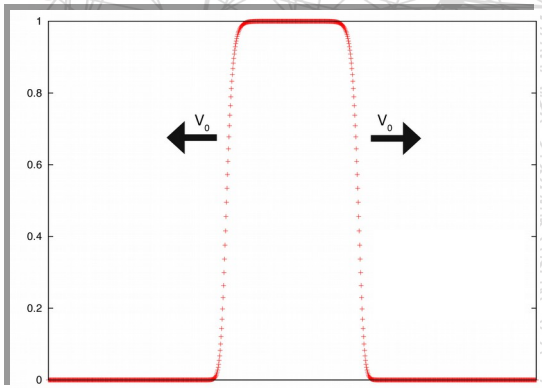
with probability α



FKPP reaction term

Reaction Diffusion problem

$$\frac{\partial}{\partial t}\theta(x,t)=D\frac{\partial^2}{\partial x^2}\theta(x,t)+\alpha\theta(x,t)(1-\theta(x,t))$$



v_0 asymptotic front velocity

$$v_0=2\sqrt{D\alpha F'(0)}$$

(localized initial condition)

$$\theta(x,t)\rightarrow 1\ \forall x$$

$$\Downarrow$$

$$M(t)=\frac{1}{L}\int_L \theta(x,t)dx$$

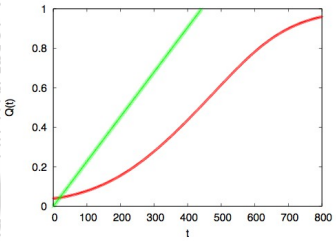
$$\Downarrow$$

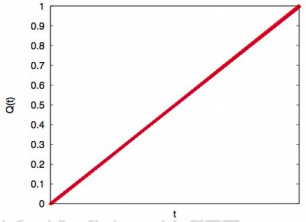
$$\lim_{t\rightarrow\infty} M(t)=1$$

$$\Downarrow$$

$$M(t)\sim 2v_0 t$$

Small size $\Longrightarrow$ Filling process





Large size $\Longrightarrow$ Front propagation

Graphs

Biological network

Neural network

Internet

Food web

World Wide Web

Infrastructures

Social network

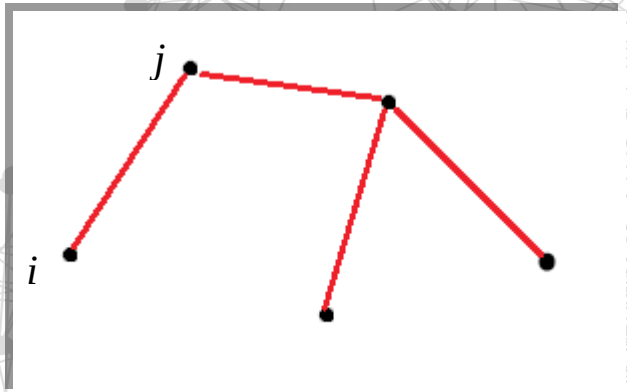
Diseases transmission

- Topological properties studied with statistical methods for large size networks

- Dynamical processes on graphs

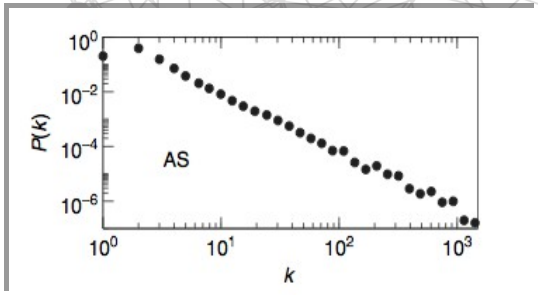
Which is the relation?

Graphs



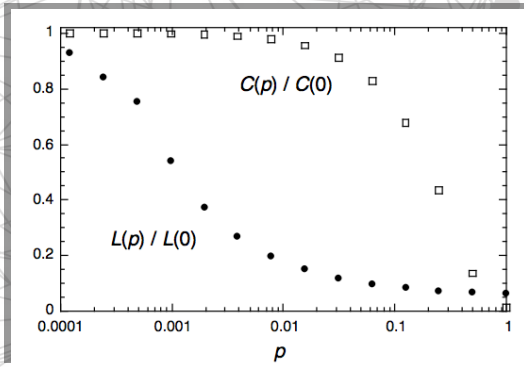
- N sites $i \in V$
- A_{ij} adjancency matrix
- M link $(i, j) \in E$
- k_i degree of connectivity
- l_{ij} Shortest path from i to j

$P(k)$ connectivity degree distribution



Average minimum distance $\langle L \rangle$

Average clustering coefficient $\langle C \rangle$



Graphs

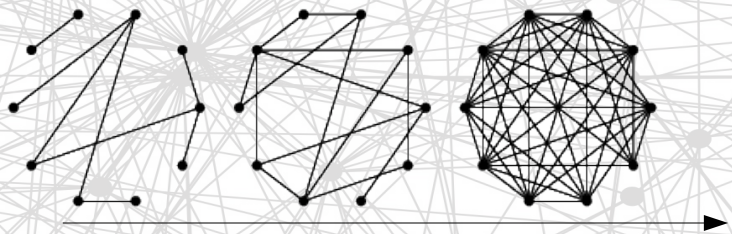
$$k_{nn,i} = \frac{1}{k_i} \sum_{j \in E(i)} k_j$$

$$k_{nn}(k) = \frac{1}{N_k} \sum_{i|k_i=k} k_{nn,i}$$

$$r = \frac{\sum_k (k - \langle k \rangle)(k_{nn}(k) - \langle k \rangle) P_l(k)}{\sum_j (j - \langle j \rangle)^2 P_l(j)}$$

Pearson assortativity coefficient

Random graphs



Connections between sites with probability p

Erdös Renyi graph (1960)

$$p_c = \ln \frac{N}{N}$$

$p \sim p_c \longrightarrow$ connected graph

$$P(k) = e^{\langle k \rangle} \frac{\langle k \rangle}{k!}$$

Poisson distribution

$$\langle C \rangle = p = \frac{\langle k \rangle}{N}$$

$$\langle l \rangle \simeq \frac{\ln(N)}{\ln \langle k \rangle}$$

Toward real networks

Small-World property

⇩

Random graphs model

⇩

Homogenous

Scale-Free network

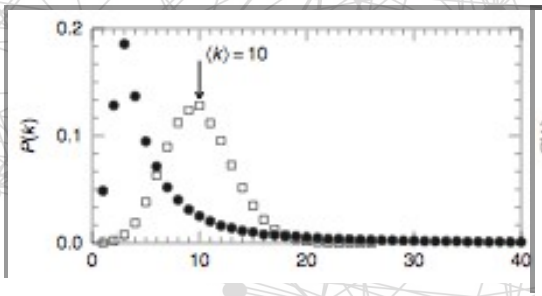
⇩

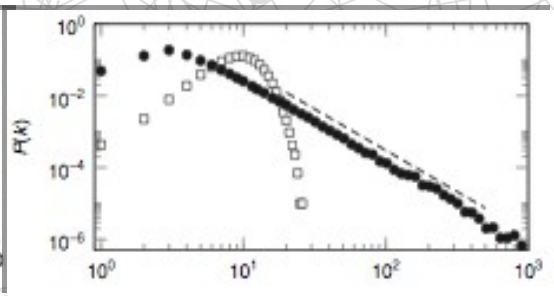
$\langle k^2 \rangle \gg 1$

⇩

Heterogenous networks

$$P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$$





$$P(k) = k^{-\gamma}$$

Reaction diffusion on graphs

$$\frac{\partial}{\partial t}\theta_i(t) = w\sum_j L_{ij}\theta_j(t) + \theta_i(t)(1-\theta_i(t))$$

$$w = \frac{D}{\alpha}$$

$$L_{ij} = k_i\delta_{ij} - A_{ij}$$

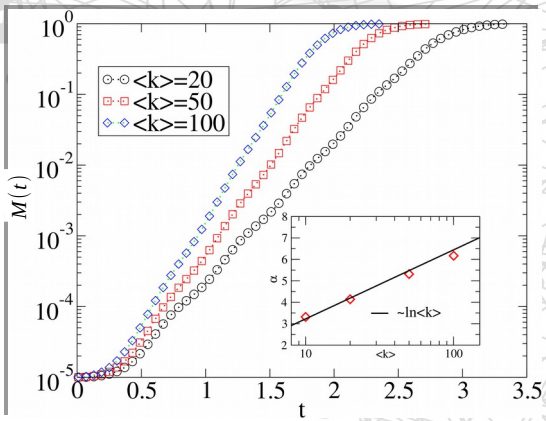
Many researchers have focused on the dynamics of processes taking place in networks, in some cases trying to understand the relation between dynamics and topological properties

[1]

Assortative network → Slower spreading

Disassortative network → Faster spreading

[2]



RD in Erdös Renyi graph

$N(l) \sim l^{d_l}$ (in graphs) → $N(l) = e^{cl}$ $(d_l = \infty)$ → $M(t) \sim e^{\alpha t}$

$$\alpha \sim \ln\langle k\rangle$$

[1]Schlapfer, Buzna - Decelerated spreading in degree-correlated networks - Phys. Rev. E 85, (2012)

[2]Burioni,Chibbaro, Vergni, Vulpiani - Reaction spreading on graphs - Phys. Rev. E 86, (2012)

Dual structured graphs

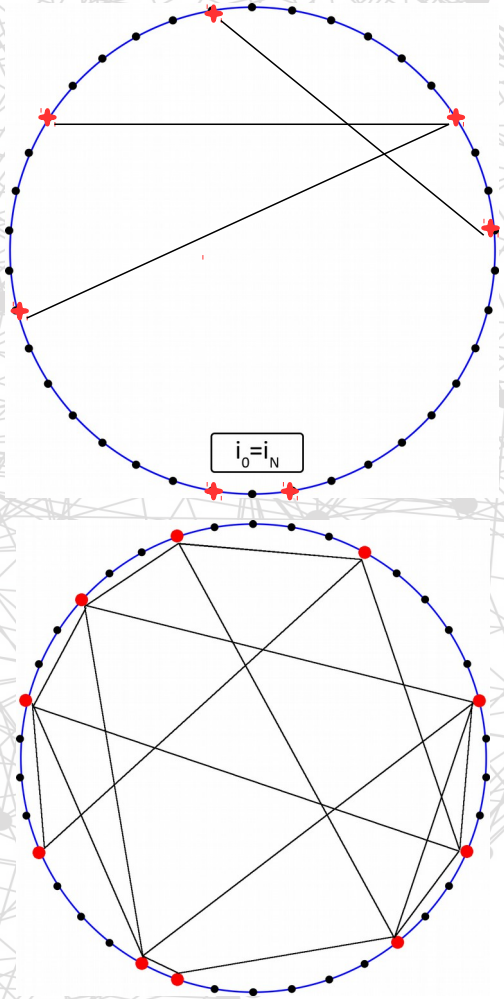
Local structure

Graph with M sites with nearest neighbour connections

N sites randomly chosen (N<M)

A type

Randomly chosen connections among the N sites with probability p



A type

Dual structured graphs

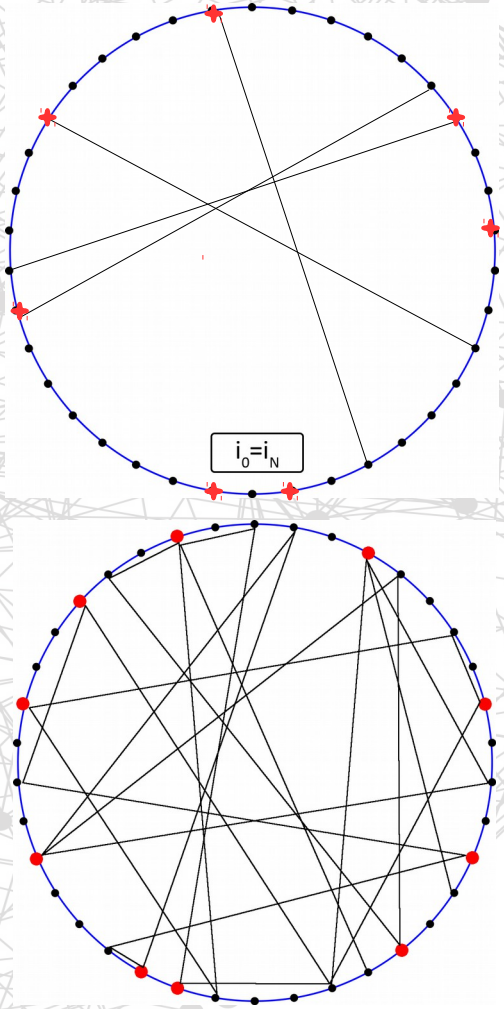
Local structure

Graph with M sites with nearest neighbour connections

N sites randomly chosen (N<M)

B type

Randomly chosen connections between the N sites and the remaining M-N sites with probability q



B type

Dual structured graphs

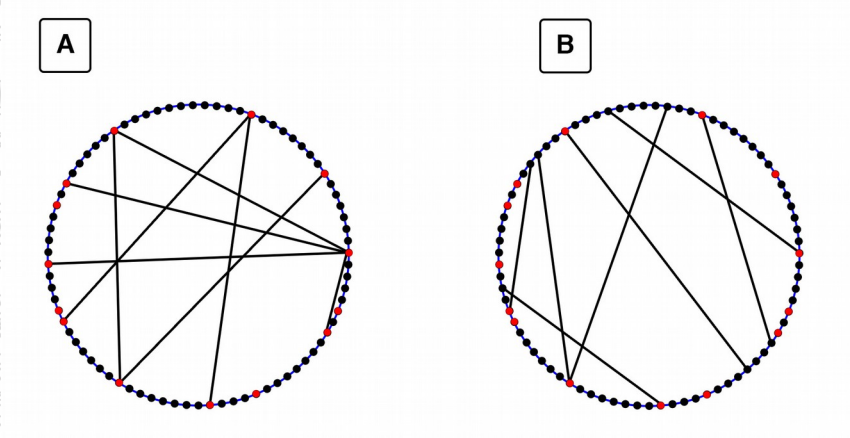
$M/N = 10^{-2}$

$P = 2\,p_c$

Assortativity

$r_A \sim 0.83$

$r_B \sim 0.04$

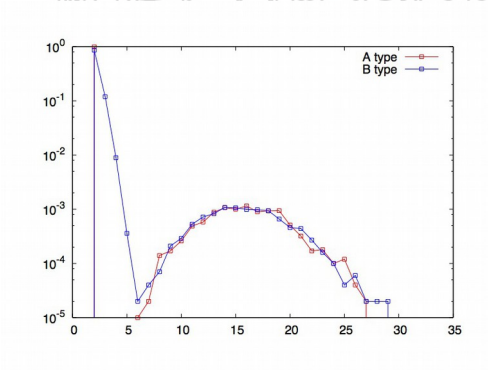


Graphs with two-folded structure of connections

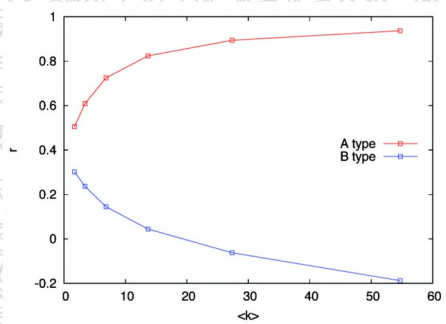
(epidemilogical diffusion inside infrastructure network, protein folding structure...)

$$\langle k \rangle^{a}_{ran} = p(N-1) = q(M-N) = \langle k \rangle^{b}_{ran} \Longrightarrow \boxed{q = \frac{p(N-1)}{M-N}}$$

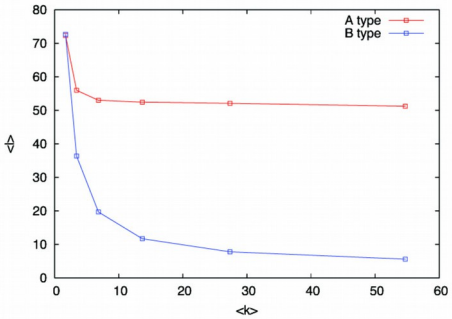
$P(k)$



$r\langle k \rangle$

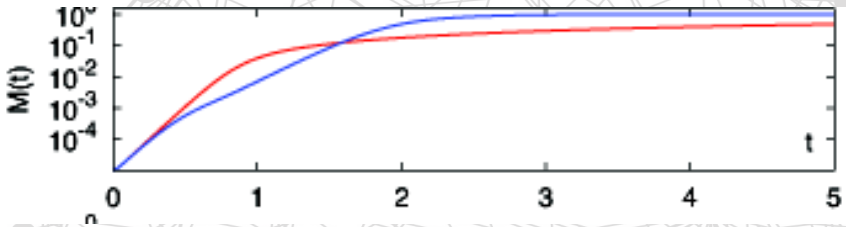


$\langle l \langle k \rangle \rangle$

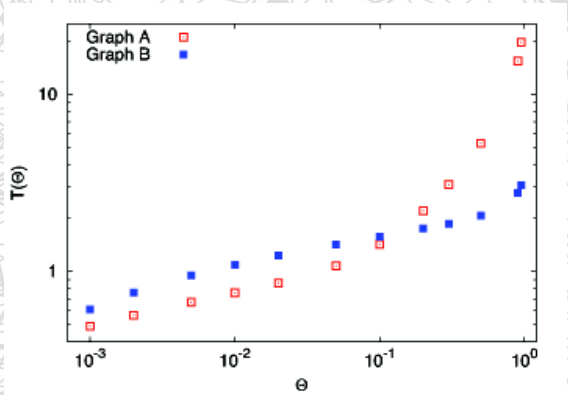


RD in dual structured graphs

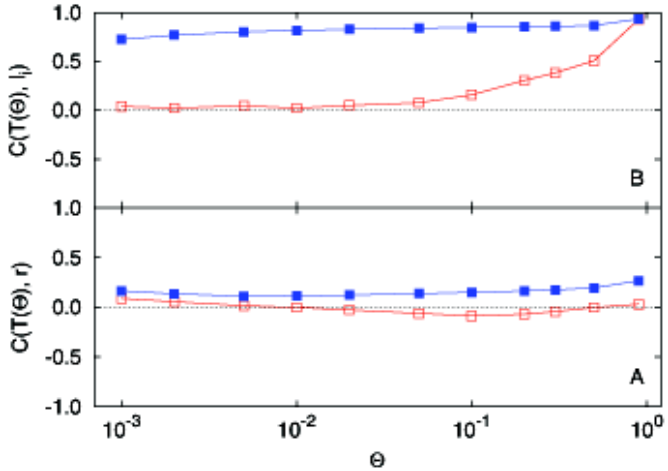
$$M^s(t)=\sum_i \theta_i(t) \quad M(t)=\frac{1}{n_s}\sum_{s=1}^{n_s} M^s(t)$$



$$T^s(\Theta)=min\{t|M^s(t)\geq\Theta\} \quad T(\Theta)=\frac{1}{n_s}\sum_{s=1}^{n_s} T^s(\Theta)$$



From a first insight we have agreement with [1] (more assortative slower spreading)



$$C(T(\Theta),x)=\frac{1}{n_s}\sum_{s=1}^{n_s}\frac{(T^s(\Theta)-T(\Theta))(x^s-x)}{\sigma_{T(\Theta)}\sigma_x}$$

The average minimum distance have a better correlation with large filling time of the system

RD in dual structured graphs

Ring distance strucuture

$$R_i(l)=\{j\in V|l_{ij}=l\}$$

Cumulative ring distance

$$G_i(l)=\{j\in V|l_{ij}\leqslant l\}$$

Percentage up to l

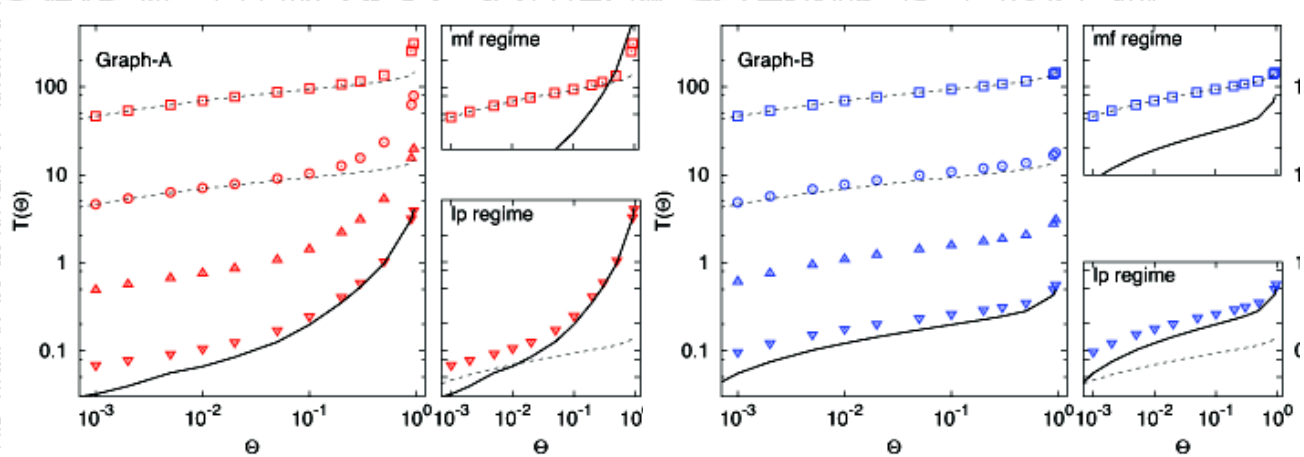
$$S_i(l)=\frac{|G_i(l)|}{M}$$

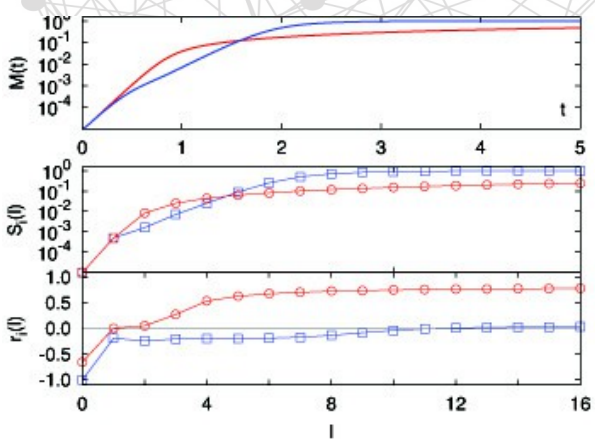
Filling time up to l

$$T_i(l)=\frac{S_i(l)}{v_0}$$

$$T_{mf}(\Theta)=\frac{1}{\alpha}\log\frac{\Theta(1-M(0))}{M(0)(1-\Theta)}$$

$$T_{lp}\sim\frac{\Theta}{\sqrt{\alpha}}$$





$$r_i(l)=\frac{\sum_{k\in G_i(l)}(k-\langle k\rangle)(k_{nn}(k)-\langle k\rangle)P_l(k)}{\sum_{j\in G_i(l)}(j-\langle j\rangle)^2P_l(j)}$$

Ring assortativity

Perspectives

- Understand the relation between topological properties of the graphs and dynamical behaviour.

- Study the problem in more realistic situations, using different reacting terms, and building real network structures. For instance in network as real transport infrastructures connections, or introducing RD epidemiological model, like SIR or SIS [3].

- Study RD problems on graphs with an agent based approach [4], passing from pde equations to stochastic processes, stressing effects of RD on a discrete population, or to analyze populations made of a few number of elements.

[3]Y. Moreno, R. Pastor-Satorras, and A. Vespignani, EPJ B-Condensed Matter and Complex Systems 26, 521 (2002)
[4]S. Berti, M. Cencini, D. Vergni, and A. Vulpiani, Physical Review E 92, 012722 (2015).