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On the optimal use of the Bethe approximation for models on graphs with loops

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Among mean field approximations the Bethe approximation (BA)[1] has attracted a renewed attention in the last two decades. Under the name of cavity method[10,12] it was succesfully applied to many different problems, such as spin glasses, error correcting codes and constraint satisfaction problems[11].

The Bethe approximation is exact for models defined on trees, and most of the interesting results were obtained in models defined on locally tree-like graphs, e.g. random graphs where loops mean size diverges in the thermodynamic limit. Beyond random graphs, i.e. on graphs with short loops, very little is known about how good is the BA. In recent years some efforts has been done for computing the finite size corrections to disordered systems defined on locally tree-like graphs[3,9] and it has been found that these corrections are related to the density of short loops. However in the general case it is unclear how wide is the application range of the BA, and in particular whether it can be of any practical use for models defined on regular lattices. We strongly believe that BA can provide some interesting insight also on models where it is expected not to be exact.

On the algorithmic side, it is worth noting that the BA self-consistency equations can be conveniently solved via an iterative algorithm called Belief Propagation (BP)[15], which is very fast and efficient in many practical applications (and thus often used even in absence of any proof of convergence or exactness). However the presence of short loops is known to be one of the main limitations to a wider use of the BA and the BP algorithm.

As a first step in my Ph.D. project I plan to apply the BA to the zero temperature random field Ising model (RFIM)[13]. Only recently it was shown[2] that the global minimum of the Bethe free energy is the maximum likelihood solution of the RFIM, irrespective of the graph on which is defined on (so even on graphs with loops). This motivated us to perform a detailed analysis of the performance of the BP algorithm, whose fixed points correspond to the minima of the Bethe free energy.

This model is one of the simplest prototypes of disordered systems with quenched disorder and still presents many unanswered questions. Even if the presence of replica symmetry breaking scenario was excluded only recently[7], the failure of dimensional reduction[14] is still an open problem, and some aspects of the phase transition still need to be clarified, as well as the anomalous slow dynamics. Renormalization group arguments show that the model can be studied in the zero temperature limit. This represent a great advantage since at $T = 0$ an exact algorithm exists with polynomial time complexity, the min-cut/max-flow algorithm[4], thus providing a benchmark for numerical studies.

We plan to consider a RFIM defined on two very different topologies: (i) random regular graphs (RRG), i.e. an ensemble of locally tree like graphs with fixed connectivity/degree, and (ii) 3-dimensional lattices. On RRG the BA is expected to be exact in the thermodynamic limit, but the presence of loops in finite graphs could induce some non-perturbative effects, while the 3D case is expected to be a much worst case scenario, due to the presence of many short loops.

At present we have some preliminary results suggesting it is worth using the BA even on models with short loops:

- We wrote down and solved numerically (via BP) the BA self-consistency equations. We found by simple inequalities that two initial conditions exist that provide maximal solutions, the ones with largest and smallest magnetization. Every fixed point configuration is bounded by these maximal solutions. Moreover we found that this two extremal configurations coincide with the lower and upper branches of the hysteresis curve, thus providing a promising connection between constrained thermodynamics and out-of-equilibrium dynamics in the RFIM.

- In the critical region we found a percolation phenomenon for the unfrozen variables, i.e. those variables taking different values in the two extremal configurations. We located the percolation threshold of the unfrozen variables and found that it coincides with the thermodynamic phase transition on the RRG ensemble, while in the 3D case it precedes the phase transition. Interesting enough the largest computational complexity is found close to the percolation threshold, rather than at the thermodynamical critical point. We believe this can have direct consequences on the RFIM dynamics.

- We designed a modified BP algorithm which is able to identify many different local minima of the Bethe free energy, by initializing BP with a convex combination of the already found fixed points. We observed that the number of stable BP fixed points (local minima of the Bethe free-energy) increases with the system size at the critical point. The algorithm is always convergent and every configuration is reached in almost linear time. An analysis of the structure of these stable states seems to suggests the possibility to work out a reweighted mean field theory, that until now was tested only in the fully connected topology[8].

- One of the best known heuristics for the spin glass (SG) optimization problem[4,5] consists in exactly optimizing a subset of nodes that form an unfrustrated cluster with the min-cut algorithm. Compared to the min-cut algorithm, the BP algorithm (although not provably exact in presence of loops) is very fast and effective: e.g., it can easily found the ground state even in the presence of a small amount of frustration. Optimizing clusters with a controlled number of frustrated links seems to be a much more powerful tool to find the global optimum. This could be a great application of the loopy BP, since at present numerical simulations of very large systems seem to be the only way for solving some open questions on the SG nature.

Therefore the aim of my Ph.D. will be to characterize the Bethe approximation, going beyond the standard field of its application. Indeed, as preliminary results suggest, much information can be extracted both from analytical and computational grounds, even on models defined on graphs with loops. A standard application will be the development of new algorithms for solving computationally hard problem (e.g. constraint satisfaction problems) via modified BP algorithms. Our main interest will be on disordered Ising spin models, but hopefully we will be able to extend the present study to the case of continuous spin variables, which may find an even broader field of application, for example in solving maximum likelihood problems via convex optimization[6].

[1] H. A. Bethe. Statistical Theory of Superlattices. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 150(871):552–575, jul 1935.

[2] Michael Chertkov. Exactness of belief propagation for some graphical models with loops. *Journal of Statistical Mechanics: Theory and Experiment*, 2008(10):P10016, oct 2008.

[3] U. Ferrari, C. Lucibello, F. Morone, G. Parisi, F. Ricci-Tersenghi, and T. Rizzo. Finite-size corrections to disordered systems on Erdös-Rényi random graphs. *Physical Review B*, 88(18):184201, nov 2013.

[4] AK Hartmann and H Rieger. *Optimization algorithms in physics.* Wiley, 2002.

[5] AK Hartmann and H Rieger. *New optimization algorithms in physics.* Wiley, 2006.

[6] Adel Javanmard, Andrea Montanari, and Federico Ricci-Tersenghi. Phase Transitions in Semidefinite Relaxations. page 71, nov 2015.

[7] F Krzakala, F Ricci-Tersenghi, and L Zdeborová. Elusive spin-glass phase in the random field Ising model. *Physical review letters*, 2010.

[8] D Lancaster, E Marinari, and G Parisi. Weighted mean-field theory for the random field Ising model. *Journal of Physics A: Mathematical and General*, 28(14):3959–3973, 1995.

[9] C Lucibello, F Morone, G Parisi, F Ricci-Tersenghi, and Tommaso Rizzo. Finite-size corrections to disordered Ising models on random regular graphs. *Physical review. E, Statistical, nonlinear, and soft matter physics*, 90(1):012146, jul 2014.

[10] M. Mézard and G. Parisi. The Bethe lattice spin glass revisited. *The European Physical Journal B*, 20(2):217–233, mar 2001.

[11] Marc Mézard and Andrea Montanari. *Information, Physics, and Computation.* Oxford University Press, 2009.

[12] Marc Mézard and Giorgio Parisi. The Cavity Method at Zero Temperature. *Journal of Statistical Physics*, 111(1-2):1–34, 2003.

[13] T. Nattermann. Theory of the Random Field Ising Model. In *Spin Glasses and Random Fields, ed. by A. P. Young, World Scientific.* 1997.

[14] G. Parisi and N. Sourlas. Random Magnetic Fields, Supersymmetry, and Negative Dimensions. *Physical Review Letters*, 43(11):744–745, sep 1979.

[15] J.S. Yedidia, W.T. Freeman, and Y. Weiss. Constructing Free-Energy Approximations and Generalized Belief Propagation Algorithms. *IEEE Transactions on Information Theory*, 51(7):2282–2312, jul 2005.