Beyond mean field in finite connectivity spin glasses

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Abstract

In this short paper I will give a description of the research topic that will be developed during my three-years Ph.D. program. The paper is divided into two sections, one for each topic proposed. Both of them rely on characterization of the low temperature phase of the Bethe lattice spin glass (BLSG). The first will focus on the zero temperature transition in magnetic field, outlining the features of such phenomenon. The second section will focus on a possible Taylor expansion for the state distribution in the spin glass phase, at any fixed temperature in null external field.

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1 Introduction

When we use the term spin glass, we refer to a particular class of spin models where the interactions between the nodes have both a ferromagnetic and antiferromagnetic support. In a few words we can say that a spin glass is the simplest model of competing systems (opposed to imitative systems, such as the Ising ferromagnetic model). Born to describe some peculiar aspects of a restricted class of magnetic alloys, spin glass models today are widely studied in a range of different interdisciplinary fields. The most common example is their wide use in optimization theory: it is possible to map a satisfiability problem to the problem of searching the ground state of a particular configuration of spin glass. Nonetheless, spin glasses' phenomenology allowed scientists to enhance the understanding of some complex behaviour in physical systems, which range from real glasses, to protein folding, opinion diffusion, neural networks and more) [15] [22]. In this work I will focus on a particular type of glass, the Bethe lattice Ising spin glass. A correct definition of the model and its peculiarities will be given in the next section. Aim of the research is to make light on the transition from ordered to glassy phase and viceversa in such model.

2 Background of the model considered

The model considered in this work is called Bethe lattice Ising spin glass (BLISG, or BLSG). It consist of a set of binary variables, $\sigma \in \{-1, +1\}$, which interact in couples with a geometry given by a Bethe lattice[1]. The Bethe lattice is defined as a lattice in which each site has the same number of neighbors and there are no loops in the system, or alternatively we may say that it is a lattice with the geometry of a tree. Such a lattice is only obtained in the thermodynamic limit. The Hamiltonian of the model is given by

$$H = \sum_{i,k} J_{ik}\sigma_i\sigma_k + \sum_i h_i\sigma_i \tag{1}$$

where J is nonzero only for those couples linked by the Bethe lattice structure. The probability distribution for J_{ik} is given by

$$P(J) = \frac{1}{2}\delta(J-1) + \frac{1}{2}\delta(J+1). \tag{2}$$

An equivalent to replica symmetric solution, known as Bethe Peierl's solution, is easy to achieve in terms of local fields. If we define the message m_{i0} passing from spin i to spin 0 as

$$m_{i0} = \frac{1}{\beta}\tanh^{-1}(\tanh(\beta J_{i0})\tanh(\beta h_i)) \tag{3}$$

we can formally write a solution for the probability distribution for the local field h_0 in the centre of the system [12]

$$Q_0(h_0) = \int \prod_i dQ_i(h_i)\delta(h_0 - \sum_i m_{i0}) \tag{4}$$

We are able to write down this equation in closed form because the Bethe lattice has no closed loops, so the spins that pass the message to spin 0 (usually referred as the cavity spin) are uncorrelated, at least before the insertion of the cavity spin. If the insertion of the cavity spin does not modify the form of the probability distributions of it's neighbouring spins, the equation written above is correct. This equation allows to write down a numerical algorithm to achieve the equilibrium probability distribution for each site of the system. The mentioned technique is well known in literature, and is often referred as Belief Propagation.

This solution gives correct results in the para/ferro phase of the system, where it is well known that it exibiths a replica symmetric behaviour. In the spin glass phase the Bethe Peierls' solution fails to return correct results [12]. This is due to the falling of the assumptions that spins in the system are uncorrelated after the insertion of the cavity spins. In order to write down a more efficient solution we have to take into account the effect of correlations after the cavity spin is inserted. This has been done in [**?**] we will report the closed equation for the probability distribution. As before, this allows to write down a numerical algorithm to find the equilibrium distribution[12].

$$Q_0^{1RSB}(h_0) = \int \prod_i dQ_i(h_i) \delta(h_0 - \sum_i m_{i0}) \exp(\beta x \Delta F(h, J)) \tag{5}$$

Where ΔF is called free energy shift, the arguments of the function are all the fields and connections involved in the procedure (the cavity spin, the neighbors and the external field acting on the cavity spin) and is given by [12]

$$-\beta \Delta F(h, J) = \log(2 \cosh \beta h_0) + \sum_i \frac{\cosh(\beta J_{i0})}{\cosh(\beta h_i)} \tag{6}$$

The value of x in equation 5 has to be selected as the one that maximizes the value of the free energy of the system.

Once this result has been obtained, it is possible to numerically execute a computation valid at each step of RSB. This has been done in my master class graduation work up to two steps of RSB and is essentially identical to the procedure performed at one step, but now the maximization has to be performed with respect to a set of two parameters, (x_1, x_2). When we go up with the number of RSB steps, the number of parameters increases. Unfortunately, due to increasing request of memory resources and the increasing computational cost of the algorithm, going more deep than a few steps of RSB is prohibitive.

3 Aim of the current work

The current work is developing in two complementary directions and both will be discussed: the first will concentrate on the behaviour of the model in magnetic field, both random gaussian and uniform, while the second part will focus on the RSB properties in zero magnetic field. While the first part is, for the moment, only numerical, the second one has a both a numerical section and an analytical one.

3.1 Characterization of the transition in zero temperature

Aim of this first section is to understand the properties of the system when it approaches the transition line in zero temperature, with the external field use as control parameter for the transition. When we have to add the external field in the system we simply have to add a neighbor in the factor graph, with a message coming from the external field. Since the use of the hyperbolic tangent, a possible numerical approach at very low, eventually zero, temperatures will have to take into account roundup errors. We do this performing the limit to $\beta \to \infty$ in equation 3 and in equation 6. The operation is easy to perform and gives back the following

$$\lim_{\beta \to \infty} \frac{1}{\beta} \tanh^{-1}(\tanh(\beta A) \tanh(\beta B)) = \operatorname{sgn} A \operatorname{sgn} B \min(|A|, |B|) \tag{7}$$

$$\lim_{\beta \to \infty} -\beta \Delta F(h, J) = |h_0| + \sum_i C(J_{i0}, h_i) \tag{8}$$

where C is a function that takes value zero if $|h| > |J|$ and ∞ otherwise.

Detect of the transition In order to detect the transition in external field, a numerical simulation valid at one-step RSB has been imlpemented. If we are in a para/ferromagnetic sector of the system (we both include paramagnetism and ferromagnetism, since the behaviour depends on which type of external field we are applying on the system: if the field is random Gaussian we obtain paramegnetism, since the values of the spins are forced to follow a Gaussian distributed random field; if the field is uniform, at high enough values of the field, all the spins are aligned, no matter the interaction between them), replica symmetric and one-step RSB computations will be identical. Down in the spin glass phase, the local field distributions in the one-RSB level will spread out, and this will be used to track down the upcoming of the transition.

Preliminary computations suggest that when we approach the critical field, h_c, not every site reacts in the same way: whilst some instantly spread out the distributions, some others will remain in a local RS phase, with a slower reaction to the approach of the RSB phase.

The main goal here is to provide a quantitative evaluation of the scaling of the number of fast reacting sites with the distance of the control parameter from the critical value. That is, given the value $\delta h = h_c - h$, we are evaluating the scaling

$$\mathrm{Prob}(\mathrm{var}(Q(h)) \neq 0) \sim \delta h^{\alpha} \tag{9}$$

and we are basically evaluating the value of α for small values of δh.

A small evidence of bubble formations In this small section I will provide a few evidences of the updescribed phenomenon.

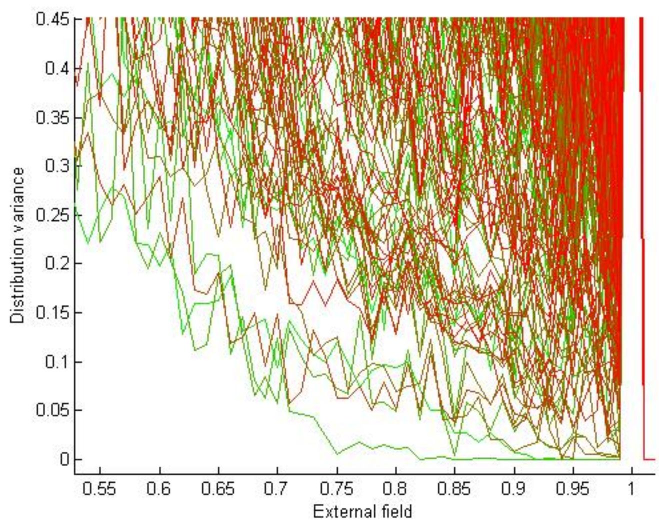

Figure 1: A system of 300 sites, coloured by label index, approaching the transition from upwards. We can see that some of the sites react slower to the approaching of RSB.

3.2 Small 2 step RSB expansion

The second investigation I am currently performing is to detect if it is possible to obtain the two-step RSB local field distribution as a perturbation of the one-step RSB distribution, given that x_1 and x_2 are close. Numerical works performed so far suggest that, once these two values are close, the two-step RSB distribution of the local fields spreads Gaussian near the one-step RSB value. The goal of this work is to search a method to avoid the high computational costs requested by multiple-steps RSB algorithms. The proceedings has gone and will go on as follows

Numerical hints of Gaussianity First of all, we made sure that the Gaussianity proposed is essentially correct. In order to do this, a two-step RSB run has been done, with values $x_1 = 0.25$ and $x_2 = 0.26$, at a temperature $T = 0.8$ in null external field.

The shape of the two-step RSB distribution returned is depicted in the following picture.

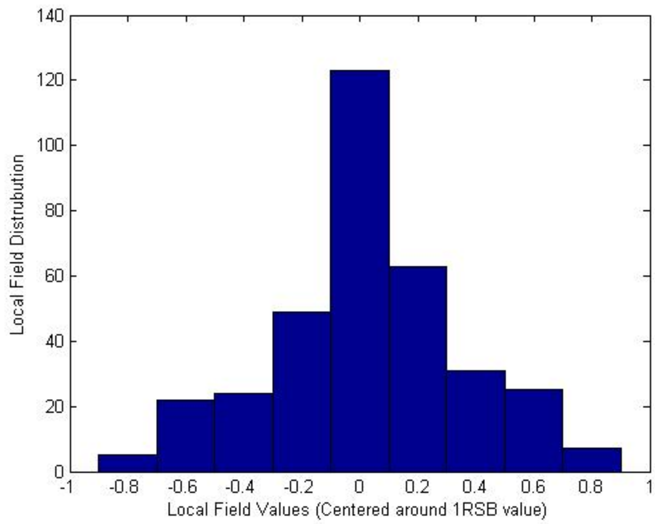

Figure 2: We can see that the distribution of local fields in the 2RSB scheme with low δx is essentially Gaussian, the picture refers to a single site distribution (site 0)

Scaling with δx and x The next computation that will be performed is to provide a quantitative value of the variance σ of the distribution. Since this essentially depends on the δx value (possibly also x itself), we are now performing a scaling of δx, in order to provide a scaling relation similar to the one presented above:

$$\sigma \sim \delta x^{\gamma(x)} \tag{10}$$

and we are searching the correct value of γ.

Improvements The final goal of this work on Gaussian expansion is to search a way to perform full RSB computations, assuming a priori that, when we add a large number of RSB steps, all the values of the x's will be very close one each other (we are putting a huge number of parameters in the $[0, 1]$ interval, this is reasonable since we know that the infinite RSB overlap function, $q(x)$, is smooth, except maybe near $x = 0$). If this is the case, the candidate is confident that it is possible to obtain the full-RSB solution for the system performing an inifinity of very small "Gaussian" steps, which will dramatically semplify the approach to a correct solution of the static of the Bethe lattice spin glass. We remark that now, the possibility to obtain an analytical full-RSB solution is out of reaching.

4 One cluster algorithm for each computation

Since all the work basically relies on the same operations, performed in different $(\beta, h_e xt)$ sectors, at different RSB steps, with different goals and outputs, the candidate has written a C++ program able to perform all these computations, which is flexible to the requests of the computation and has a personalizable output, both in run and at finished computation. The candidate believes that such a routine will be proficient in answering a lot of his questions on the behaviour of the BLSG. Such algorithm is ideated to run in cluster computers and is written ad hoc to perform parallel computing. Written using the oMPI libraries, will be distributed online under gnu-gpl once a graphical suite has been implemented. In this section we recall the flux diagram of the algorithm.

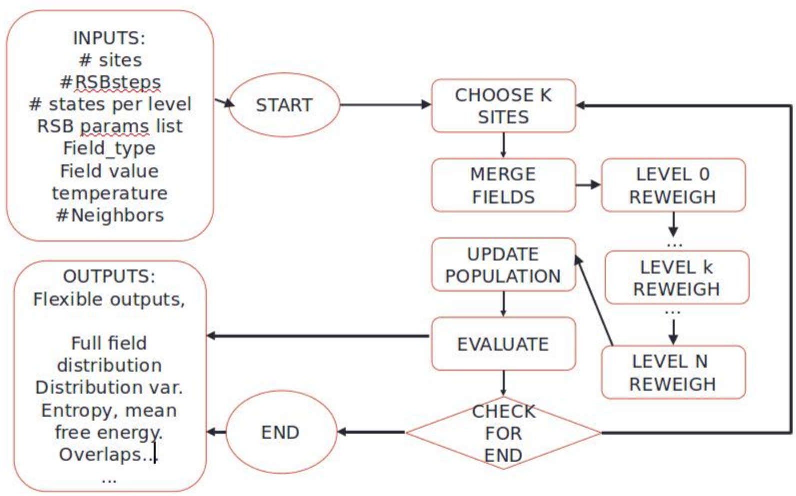

Figure 3: The flux diagram for the general purpose BLSG algorithm set up, the algorithm may run at every temperature and external field, when the temperature goes under the treshold of 0.1*, the algorithm runs a protected version, avoiding the possibility of roundup errors.*

5 Conclusion

In this paper the candidate presented his proceedings on his main research topic, the Bethe lattice spin glass. The work on BLSG is divided in two subsubjects, one willing to investigate the behaviour of the system in the zero-temperature spin glass phase, the other focusing on the behaviour of BLSG in the zero-field, finite temperature spin glass phase. The work is a straightforward continuation of what has been made under the supervisor's supervision during the master graduation work.

Among with the topics under study, an all purpose algorithm, valid for every topic under investigation, has been presented.

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