Critical properties of disordered XY model on sparse random graphs

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PhD in Physics
XXIX cycle
1 Discrete and Continuous Spin Glass Models

2 XY and clock model

3 Small fluctuations of the XY model

4 Conclusions
What a spin glass is

It is a magnet in which spins are not aligned along a regular pattern, due to the presence of some kind of quenched disorder:

\[ \mathcal{H}[\{\sigma\}] = - \sum_{(i,j)} J_{ij} \sigma_i \sigma_j - \sum_i H_i \sigma_i , \quad \sigma = \pm 1 \]

The disorder can be given by random couplings:

\[ J_{ij} = \pm 1 \quad \text{or} \quad J_{ij} \sim \text{Gauss}(J_0, \bar{J}) \]

or by an external random field:

\[ H_i = \pm 1 \quad \text{or} \quad H_i \sim \text{Gauss}(0, \bar{H}) \]
Mean-field approach to spin glasses (I)

Most of knowledge about spin glasses comes from mean-field approaches, given by suitable topologies:

- Fully connected graphs
- Sparse random graphs (Bethe lattices)
Mean-field approach to spin glasses (II)

Different techniques and results:

**Fully connected graphs**

**PROS:**
- exact analytical results, though “trivial”
- Gaussian couplings $J_{ij}$

**CONS:**
- infinite average degree $\bar{z}$ (not physical!)
- all sites are equivalent

**Sparse random graphs (Bethe lattices)**

**PROS:**
- finite average degree $\bar{z}$
- notion of distance
- spatial heterogeneity
- closer to finite dimensional lattices

**CONS:**
- more involved analytical approach
- approx. numerical results
SK and other spin glass models

When $\sigma_i = \pm 1$ on a fully-connected graph with Gaussian couplings we have the SK model, exactly solved through a very refined analytical technique (replica trick + replica symmetry breaking) by G. Parisi.

Other spin glass models with discrete variables (e.g. Potts, colouring, ...) can be solved in an analogous manner. So after 40 years we know (almost) everything about spin glass models with discrete variables:

- exact and very refined analytical studies
- very high precision numerical results

But what about models with vector spins (i.e. continuous variables)? So far there are very few studies about them!

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1D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. 35 (1975) 1792
2M. Mézard et al, Spin Glass Theory and Beyond (1987)
About vector spin glasses

\[ H[\{\vec{\sigma}\}] = - \sum_{(i,j)} \vec{\sigma}_i \cdot \hat{U}(\omega_{ij}) \cdot \vec{\sigma}_j - \sum_i \vec{H}_i \cdot \vec{\sigma}_i, \quad |\vec{\sigma}|^2 = 1 \]

Why is it so difficult to deal with these models?

- prob. density functions instead of discrete prob. distributions
- presence of continuous symmetries (⇒ Goldstone theorem, …)
- presence of zero-energy collective modes (spin waves)

But there also new features that make these models interesting:

- isotropy of spins in space
- small fluctuations at low \( T \)
- new phase transitions (Kosterlitz-Thouless, Gabay-Toulouse, …)
- applications in continuous optimization problems

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1 J. M. Kosterlitz and D. J. Thouless, J. Phys. C: Solid State Phys. 6 (1973) 1181
What we know about them

Scalar spins:
- f-RSB solution on FCGs
- 1RSB solution on RGs (very close to the exact one)
- very good understanding of optimization problems
- correlation functions rather well characterized
- random field Ising model very deeply studied

Vector spins:
- RS solution on FCGs and RGs + some hints on 1RSB solution
- no known f-RSB solution
- very few applications in optimization problems
- correlation functions and small fluctuations matrix not well characterized
- behaviour in random field not completely understood
Discrete and Continuous Spin Glass Models

XY and clock model

Small fluctuations of the XY model

Conclusions
XY model on Bethe lattices (I)

We started a systematic study of the XY model ($m = 2$ unit vector spins) on Bethe lattices, by exploiting cavity method:

$$\mathcal{H} = - \sum_{(i,j)} J_{ij} \cos (\theta_i - \theta_j) , \quad \theta \in [0, 2\pi]$$

At RS stage, each directed edge brings a message, given by marginal $\eta_{i \rightarrow j}(\theta_i)$. We can write iterative equations for these messages:

$$\eta_{i \rightarrow j}(\theta_i) = \frac{1}{\mathcal{Z}_{i \rightarrow j}} \prod_{k \in \partial_i \setminus j} \int_0^{2\pi} d\theta_k e^{\beta J_{ik} \cos (\theta_i - \theta_k)} \eta_{k \rightarrow i}(\theta_k)$$

called BP equations.

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BP equations can be solved **analytically** in the high $T$ region:

$$
\eta_{i \rightarrow j}(\theta_i) = \frac{1}{2\pi} \left[ 1 + \sum_{\ell=1}^{\infty} \left( a_{\ell}^{(k \rightarrow i)} \cos(\ell\theta_i) + b_{\ell}^{(k \rightarrow i)} \sin(\ell\theta_i) \right) \right]
$$

by using Fourier expansion.

And **numerically** in low $T$ region:

$$
\theta_i \in [0, 2\pi] \Rightarrow \theta_i \in \left\{ 0, \frac{1}{Q} 2\pi, \frac{2}{Q} 2\pi, \frac{3}{Q} 2\pi, \ldots, \frac{Q-1}{Q} 2\pi \right\}
$$

by using the $Q$-states clock model.
XY model phase diagram

Clock model with $Q = 64$ gives the $p$ vs. $T$ phase diagram of the XY model.

Disorder distribution:

$$\mathbb{P}(J_{ik}) = p \delta(J_{ik} - J) + (1 - p) \delta(J_{ik} + J)$$

Order parameters of phase transitions:

$$\vec{m} = \frac{1}{N} \sum_i \vec{m}_i , \quad q = \frac{1}{N} \sum_i |\vec{m}_i|^2$$

$$\lambda_{BP} = \lim_{t \to \infty} \log \left[ \frac{1}{N} \sum_i \| \delta \eta_i(t) \|_2^2 \right]$$

Main features:

- presence of RSB mixed phase
- RSB at $T = 0$ as soon as $p < 1$

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The Q-states clock model

Q very large should not introduce corrections.

But Q quite small modifies the continuous features of the model.

Questions:
1. What is the effect of discretization?
2. Is there any threshold in Q?
3. How do physical observables converge in Q?
4. Does the universality class change in Q? How?

First answers:
- \( Q = 2, 3, 4 \) are special cases (resp. Ising, 3-state Potts, “double” Ising).
- For \( Q \geq 5 \) it actually mimics the XY model.
$Q$-states clock phase diagram
Exponential convergence of physical observables

Disorder enhances convergence! Error is exponentially small in $Q$:

$$\log \Delta f^{(Q)} \sim A - (Q/Q^*)^\beta$$

Simple Exponential for $T > 0$

$\beta = 1$, $Q^* \simeq 2.57(1)$

Stretched Exponential for $T = 0$

$\beta = 1/2$, $Q^* \simeq 0.67(1)$

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1RSB BP equations (I)

BP equations are exact when no short loops are present:

- trees and large sparse random graphs

When short loops are present, in the RSB phases (“SG” and “M”) BP fails.

Q-states clock model and XY model have f-RSB solutions. Study of universality class needs at least a step of RSB.

In the 1RSB approach Gibbs measure breaks into an exponential number of states. In each state RS BP equations hold, then they have to be reweighed according to their free energy shift.

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1RSB BP equations (II)

Hence 1RSB BP equations basis on a two-level hierarchy of populations:

\[ P_{i \rightarrow j}[\eta_{i \rightarrow j}] = \int \left( \prod_{k \in \partial i \setminus j} \mathcal{D} \eta_{k \rightarrow i} \right) P_{k \rightarrow i}[\eta_{k \rightarrow i}] \delta \left[ \eta_{i \rightarrow j} - \mathcal{F}[\{\eta_{k \rightarrow i}\}] \right] \left( Z_{i \rightarrow j} \right)^x \]

\[ \mathcal{P}[P_{i \rightarrow j}] = \int \left( \prod_{k \in \partial i \setminus j} \mathcal{D} P_{k \rightarrow i} \right) \mathcal{P}[P_{k \rightarrow i}] \delta \left[ P_{i \rightarrow j} - \mathcal{G}[\{P_{k \rightarrow i}\}] \right] \]

where:

- \( N_{\text{distr}} \) marginals \( \eta \), described by functional distribution \( P[\eta] \)
- \( N_{\text{pop}} \) populations of marginals, described by functional distribution \( \mathcal{P}[P] \)

Parisi function \( q(x) \) (overlap) gives the universality class:

- \( Q = 2, 4: \) Ising
- \( Q = 3: \) 3-states Potts
- \( Q \geq 5: \) XY

Discrete and Continuous Spin Glass Models

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Susceptibilities in vector spin glass models

Susceptibility $\chi$ describes the response of the system to an external perturbation:

$$\chi(i,j) = \frac{\partial \langle \sigma_i \rangle}{\partial H_j} = \beta \langle \sigma_i \sigma_j \rangle_C$$

and, as usual, $\chi \equiv \sum_j \chi(i,j) \to \infty$ corresponds to a phase transition!

When $m \geq 2$, susceptibility becomes a matrix:

$$\chi_{\mu,\nu}(i,j) = \frac{\partial \langle \sigma_{i,\mu} \rangle}{\partial H_{j,\nu}} = \beta \langle \sigma_{i,\mu} \sigma_{j,\nu} \rangle_C$$

and we have two different kinds of susceptibilities:

- a longitudinal one, $\chi_l$
- a transverse one, $\chi_t$
Small fluctuations of the XY model

Spontaneous symmetry breaking

In ordered ferromagnets at $H = 0$:

- $T = T_c$ (phase transition)
  \[ \chi_l = \infty, \quad \chi_t = \infty \]
- $T < T_c$ (Goldstone modes)
  \[ \chi_l < \infty, \quad \chi_t = \infty \]

When $H \neq 0$ no longer Goldstone modes!
Symmetry is explicitly broken:

\[ \chi_t < \infty \quad \text{for} \quad T < T_c \]

In spin glasses:

- SK model: $\chi = \chi_l$ diverges at dAT line
- vector models: when do $\chi_l$ and $\chi_t$ diverge???
Gabay-Toulouse results on fully-connected graphs

When $H \neq 0$ symmetry is explicitly broken, but we still have null modes in the free energy landscape! Some $\chi$ has to diverge…

We can define $\parallel$ and $\perp$ overlaps with respect to direction $\hat{x}$ of the field:

$$q_l \equiv \frac{1}{N} \sum_i m_{i,x}^2,$$

$$q_t \equiv \frac{1}{N} \sum_i \sum_{\mu \neq x} m_{i,\mu}^2,$$

and identify two transitions:

(a): freezing of $q_t$, GT line (if uniform $\vec{H}$)
(b): freezing of $q_l$, dAT line (always)

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Our results on sparse random graphs

We exploit again SuscProp, identifying phase transitions when $\lambda_{BP} = 0$.

Both GT and dAT lines are recovered when using constant strength $|\vec{H}|$ and direction drawn from:

$$\mathcal{P}_\phi \sim \begin{cases} 
\delta(\phi) & \text{GT line} \\
\text{Unif}[0, 2\pi] & \text{dAT line}
\end{cases}$$

As expected, at $T = 0$ on sparse graphs we get finite critical values of the field.

Critical exponents are the same of FCGs:

$$\delta T_c(H) \sim H^2 \quad \text{GT line}$$

$$\delta T_c(H) \sim H^{2/3} \quad \text{dAT line}$$

\[ H_{GT} \simeq 4.82 \]

\[ H_{dAT} \simeq 1.06 \]

\[ T_c \simeq 0.487 \]
Small fluctuations of the XY model

Soft modes in the free energy landscape

Low-energy excitations are given by flatness in free energy landscape. Let’s compute its Hessian matrix $\mathbf{H}$ at $T = 0$ on the ground state configuration:

$$\mathcal{H} = -H \sum_i \cos (\theta_i - \phi_i) - \sum_{(i,j)} J_{ij} \cos (\theta_i - \theta_j) \Rightarrow \text{g.s. } \{\theta_i^*\}$$

$$\mathbf{H}_{ij} \equiv \left. \frac{\partial^2 F}{\partial \theta_i \partial \theta_j} \right|_{T=0,\{\theta_i^*\}} = \left. \frac{\partial^2 \mathcal{H}}{\partial \theta_i \partial \theta_j} \right|_{\{\theta_i^*\}} \quad [N \times N \text{ matrix}]$$

- Eigenvalues of $\mathbf{H} \Rightarrow$ spectral density, energy of $T = 0$ excitations
- Eigenvectors of $\mathbf{H} \Rightarrow$ inherent structures of free energy landscape, flat directions (if any)
The SG Random Field XY model

Zero eigenvalues not linked to RSB. No localization in eigenvectors.

For $C = 3$ RRG:
- Gap closes at $H \simeq 4$
- $dAT$ occurs at $H \simeq 1.06$

Gapless spectral density goes as:

$$ \rho(\lambda) \sim \lambda^{3/2} $$

In FCGs exponent is $1/2$.

No diverging $\chi$ in each state!
Need to average over states $\Rightarrow$ RSB!

$^1$C. Lupo, G. Parisi and F. Ricci-Tersenghi, work in progress
The Random Field XY model

RFIM shows no RSB. RFXYM does!

\[ T = 0, 0.004, 0.008, 0.012 \]

\[ \lambda_{BP} \]
Correlation functions

Connected correlation functions computed along a chain of length $r$:

$$C(r) \equiv \| \mu(\theta_0, \theta_r) - \mu(\theta_0) \mu(\theta_r) \|$$
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Conclusions

- Q-states clock model is a reliable and efficient proxy for XY model
- XY is more glassy than Ising: closer to structural glasses?
- Low-energy excitations even far from critical point
- Heterogeneous long-range correlations
- RSB behaviour of RFXYM