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A History of the Science of Light

From Galileo's telescope to the laser and the quantum information technologies

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Lecture 7: Entanglement

Entanglement is the strangest feature of quantum physics. Interacting quantum systems generally end-up in a state which cannot be described as a tensor product of independent quantum states corresponding to its parts. The non-separable global state is said to be entangled. This property, a direct consequence of the superposition principle, has been known since the early days of quantum physics as a fundamental feature of atomic states. Interactions between quantum systems or measurements performed on them generally result in entanglement. In order to describe measurements performed on a system A entangled with an unobserved system B, the concept of the quantum state vector in a Hilbert space has to be replaced by that of a density operator. The properties of this operator and its physical interpretation are described. The density operator is useful to quantify the degree of entanglement of a bi-partite quantum system. Maximum entanglement occurs when all information about the system is contained in correlations between measurements performed on the two parts and no information can be obtained from observations made on one part alone. The connection between entanglement and complementarity is discussed. Entanglement of a system with a large environment is analysed in connection with the notion of decoherence. Quantum measurement is also described as a process starting by an entanglement between the measured microscopic system and a macroscopic measuring apparatus. The principle of conditional quantum gates coupling a « control » and a « target » qubit will be studied. The properties of these gates, essential tools in quantum information as generators and analysers of entanglement, will be described. Finally, we relate entanglement to the «non-locality» of quantum physics. We recall Einstein's argument which stated, against Bohr, that the existence of correlations between entangled systems separated by large distance was the indication that quantum physics was incomplete and implied the existence of supplementary hidden variables. We present Bell's formulation of Einstein's argument and describe an experiment demonstrating the non-existence of these hidden variables, vindicating Bohr against Einstein. We also describe how two partners can exchange a quantum state by a process called « teleportation ». This is a procedure using a quantum channel (the sharing of entangled particles) and a classical one (communication of two bits by an electromagnetic signal).

Quantum Entanglement

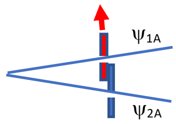
Consider a system S made of two parts A and B in Hilbert spaces H_A and H_B. Expand the most general state of S on a basis of kets tensor products $|i_A\rangle|\mu_B\rangle$

$$|\psi_S\rangle = \sum_{i,\mu} \alpha_{i,\mu} |i_A\rangle \otimes |\mu_B\rangle$$

$|\psi_S\rangle$ is said to be entangled if it cannot be factored as a tensor product of a state of A by a state of B:

$$|\psi_S\rangle \neq |\varphi_A\rangle \otimes |\phi_B\rangle$$

Entanglement is a general feature of composite systems, implied by the linearity of quantum physics. It results from interactions between A and B or from measurements of observables of O_S admitting entangled A-B states as eigenstates. We consider here the discrete dimension case, which generalizes straightforwardly to infinite dimension by replacing sums by integrals. We have already encountered entangled systems. In the Young double slit thought experiment, the system S « particle + moving slit » is entangled. The particle (A) crosses the double slit in the superposition state $|\psi_{1A}\rangle+|\psi_{2A}\rangle$, the moving slit (MS) being in ground state $|0_{MS}\rangle$ of its harmonic oscillator. If A crosses through slit 1, it receives a momentum kick -p while the slit moves with momentum p in opposite direction.



$$\frac{1}{\sqrt{2}}\left(|\psi_{1A}\rangle + |\psi_{2A}\rangle\right)\otimes|0_{MS}\rangle \rightarrow$$

$$\frac{1}{\sqrt{2}}\left(T_{-P}|\psi_{1A}\rangle \otimes |p_{MS}\rangle + |\psi_{2A}\rangle \otimes |0_{MS}\rangle\right)$$

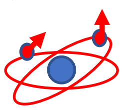
Entanglement is an essential feature at the heart of the concept of complementarity, of measurement theory and decoherence. It is also at the center of the notion of quantum non-locality. We discuss here these important aspects of quantum theory after recalling some definitions and general properties of entanglement.

Examples of entanglement in atomic systems

Two quantum systems in a pure state are entangled when this state cannot be factored as a tensor product of states belonging to the Hilbert state of each component.

$$|\psi\rangle_{AB} \neq |\psi_A\rangle \otimes |\psi_B\rangle$$

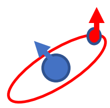
Example 1: ground state of He atom:



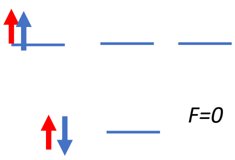
The two electrons are in the ground orbital state (n=1, L=0). The orbital wave function is symmetrical by electron exchange. The spin state must thus be the S=0 antisymmetrical state: This state is non separable, hence entangled:

$$|S=0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2\right) \neq |\psi_1\rangle \otimes |\psi_2\rangle$$

Example 2: Hyperfine structure of H ground state

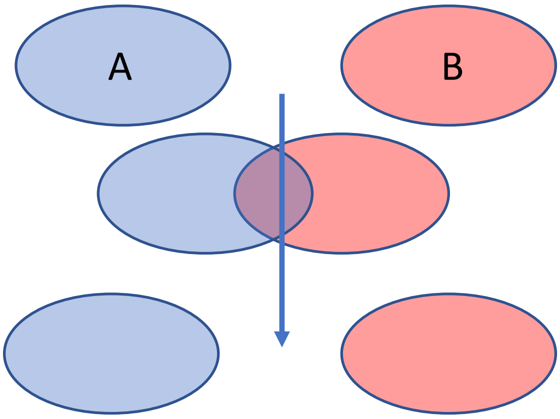


The electron spin interacts with the nuclear spin of the proton. The total spin F is either 1 one or zero. The F=0 ground state is entangled:



$$|F=0\rangle = \frac{1}{\sqrt{2}}\left(|+_p\rangle|-_e\rangle - |-_p\rangle|+_e\rangle\right)$$

Entanglement produced by interaction



$$\left|\Psi(t_i)\right\rangle_{AB} = \left|\psi(t_i)\right\rangle_A \otimes \left|\psi(t_i)\right\rangle_B$$

$$\left|\Psi(t_f)\right\rangle_{AB} \neq \left|\psi(t_f)\right\rangle_A \otimes \left|\psi(t_f)\right\rangle_B$$

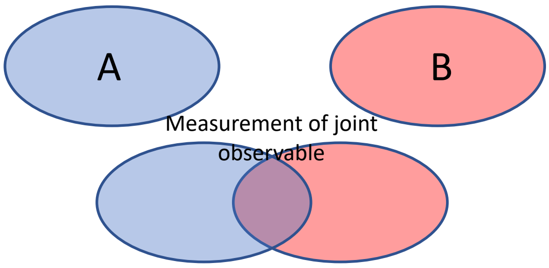
Example: flipping of two spins interacting with the Hamiltonian: $H = \varepsilon_0\left(\left|\uparrow\downarrow\right\rangle\left\langle\downarrow\uparrow\right| + \left|\downarrow\uparrow\right\rangle\left\langle\uparrow\downarrow\right|\right)$

$$i\hbar\frac{\partial\left|\psi(t)\right\rangle}{\partial t} = H\left|\psi(t)\right\rangle \qquad \left|\psi(t)\right\rangle = \cos\left(\frac{\varepsilon_0 t}{\hbar}\right)\left|\uparrow\downarrow\right\rangle - \sin\left(\frac{\varepsilon_0 t}{\hbar}\right)\left\langle\downarrow\uparrow\right|$$

At half-flipping time, maximum entanglement

$$\frac{\varepsilon_0 t}{\hbar} = \frac{\pi}{4} \qquad \left|\psi(t)\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle - \left\langle\downarrow\uparrow\right|\right)$$

Entanglement produced by joint measurement



$$\left|\Psi(t_i)\right\rangle_{AB} = \left|\psi(t_i)\right\rangle_A \otimes \left|\psi(t_i)\right\rangle_B$$

System state collapses in entangled state

$$\left|\Psi_{after}\right\rangle_{AB} \neq \left|\psi\right\rangle_A \otimes \left|\psi\right\rangle_B$$

Example: measure total spin S on two spins initially in the state $\left|\uparrow\downarrow\right\rangle$

$$\left|\uparrow\downarrow\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle\right) + \frac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle\right)$$

Result $S=1$ found with probability ½ collapses state in entangled state $\frac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle\right)$

Result $S=0$ found with probability ½ collapses state in entangled state $\frac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle\right)$

Density operator formalism

Let us now introduce the density operator formalism, very useful to describe measurements performed on the parts of an entangled system S. To each state $|\psi_S\rangle$ of S , we associate the projector:

$$|\psi_S\rangle \quad \Rightarrow \quad \rho_S = |\psi_S\rangle\langle\psi_S|$$

We call it the density operator of the global state. This operator has a (0,1) spectrum, the state $|\psi_S\rangle$ corresponding to eigenvalue 1, all the states orthogonal to $|\psi_S\rangle$ to eigenvalue 0.

Consider now an observable O_S, diagonal in a basis $\{|i,\alpha\rangle\}$ of states whose projectors are $|i,\alpha\rangle\langle i,\alpha|$:

$$O_S = \sum_i a_i P_i \qquad \text{with} \qquad P_i = \sum_\alpha |i,\alpha\rangle\langle i,\alpha|$$

According to the measurement postulate, the probability to find the result a_i when measuring O_s on state $|\psi_S\rangle$ is:

$$\Pi_i = \sum_\alpha \left|\langle\psi|i,\alpha\rangle\right|^2 = \sum_\alpha \langle i,\alpha|\psi\rangle\langle\psi|i,\alpha\rangle = T_r\left(P_i\rho_S\right)$$

We recall that the trace of an operator is the sum of its diagonal elements (result independent of basis)

Up to now, we have simply rewritten in an equivalent formalism the postulate of measurement, replacing a quantum state by its projector. This replacement becomes interesting when we focus on measurements performed on one part A or B of S, while disregarding the other part. We discuss this on next page.

Measurements on part A of a system S made of two parts A and B

Suppose that we measure O_A without looking at B. This is equivalent to measuring $O_A.I_B$ where I_B is the identity operator in the Hilbert space of B. We can now expand the O_AI_B observable over a basis of eigenstates as:

$$O_A \otimes I_B = \sum_i a_i P_i \qquad \text{with} \qquad P_i = \sum_{\beta,\mu} |i,\beta;\mu\rangle\langle i,\beta;\mu|$$

The index β accounts for possible degeneracies of the O_A spectrum and μ refers to any basis in the Hilbert space of B. The measurement postulate then gives the probability Π_i of finding result a_i when measuring O_A:

$$\Pi_i = \sum_{\alpha,\mu} \langle i,\beta;\mu|\psi\rangle\langle\psi|i,\beta;\mu\rangle = \sum_\beta \langle i,\beta|\left(\sum_\mu \langle\mu|\psi\rangle\langle\psi|\mu\rangle\right)|i,\beta\rangle$$

The bracket is the trace over B of the global density operator ρ_S: we call it the partial density operator of the subsystem A

$$\rho_A = T_{rB}\left(\rho_S\right)$$

We define in the same way

$$\rho_B = T_{rA}\left(\rho_S\right)$$

We have thus finally: $\Pi_i = T_{rA}\left(\rho_A P_{iA}\right)$ with $P_{iA} = \sum_\beta |i,\beta\rangle\langle i,\beta|$

And the expectation value of O_A is: $\langle O_A\rangle = \sum_i \Pi_i a_i = T_{rA}(\rho_A \sum_i P_{iA} a_i) = T_{rA}(\rho_A O_A)$

We retrieve the same formula as for a measurement performed on a global system, with one important difference: in general, ρ_A, and ρ_B unlike ρ_S, are not projectors, but sums of projectors (see next slide).

Properties of density operators: pure states and mixtures

The density operators of parts of a global system are Hermitian operators with positive eigenvalues and trace equal to 1. Their elements in a diagonal basis representing probabilities must be positive. The unity trace is required by the definition of probabilities:

$$\sum_i \Pi_i = T_{rA}\left(\rho_A \sum_i P_{iA}\right) = T_{rA}\rho_A = 1$$

The most general operator of this kind writes in the basis where it is diagonal:

$$\rho_A = \sum_{\lambda_i} \lambda_i |\varphi_i\rangle\langle\varphi_i| \quad ; \quad \lambda_i \geq 0 \; and \sum_i \lambda_i = 1$$

If only one λ_i is different from zero, we retrieve the pure case discussed earlier: the density operator is a projector and the system is described by a pure quantum state containing the maximum information we can get about A.

As soon as more than one λ_i is non zero, the density operator describes a « mixture of states ». To predict the outcomes of measurement, we can consider that the system A is distributed among different $|\varphi_i\rangle$ states with the probabilities λ_i This statistical uncertainty comes in addition to the uncertainty due to the randomness of quantum measurement. It is due to the fact that by getting entangled to another system, A and B considered separately have lost some information countained in the global system A+B. Note that the λ_i's are not probability amplitudes. They are positive real numbers and they are not involved in interference effects.

How recognize that a density operator represents a pure state or a mixture? Compute $Tr(\rho^2)$

Pure state: only one λ_i non zero:

$$T_r\rho_A^2 = 1$$

Mixture: more than one λ_i is non zero and $Tr\rho_A^2 < 1$

$$T_r\rho_A^2 = \sum_i \lambda_i^2 < \left(\sum_i \lambda_i\right)^2 = 1$$

Density operator of a spin ½: Bloch vector of a mixture

Let us consider as an important example the case of a spin ½. If it is part of a larger system and we are considering only measurements on it disregarding its environment, it will be described by its density operator. It is represented by a 2x2 Hermitian matrix which can always be expanded on a basis of 4 operators, the three Pauli matrices plus the identity operator:

$$\rho_A = a_0 I + \sum_i a_i \sigma_i$$

The a_0, a_i coefficients must be real to ensure that ρ is Hermitian. Moreover, the trace of the Pauli matrices being 0, we must have a_0=1/2 to ensure that the trace of ρ is 1. Finally we compute the expectation values of the Pauli operators and we find:

$$\langle\sigma_i\rangle = T_{rA}(\sigma_i \rho_A) = a_i T_r(\sigma_i^2) = 2a_i$$

(because $T_r(\sigma_i\sigma_j) = 2\delta_{ij}$)

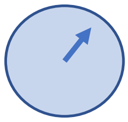
We thus get the most general density operator of a spin:

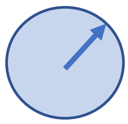
$$\rho_A = \frac{1}{2}\left[I + \vec{P}_{Bloch}.\vec{\sigma}\right] \qquad \vec{P}_{Bloch} = \sum_i \langle\sigma_i\rangle \vec{e}_i$$

$$= \frac{1}{2}\begin{vmatrix} 1+P_z & P_x + iP_y \\ P_x - iP_y & 1-P_z \end{vmatrix} \qquad T_{rA}(\rho_A)^2 = \frac{1+\vec{P}^2}{2}$$

$$T_{rA}(\rho_A)^2 \leq 1 \quad \rightarrow \quad |\vec{P}| \leq 1$$

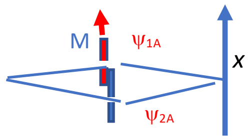
The "Bloch vector" P_{Bloch} has a modulus smaller or equal to 1. This generalizes the Bloch sphere representation of pure state (see lecture 6). Any *mixture* of spins is represented by a Bloch vector *inside* the Bloch sphere of radius one. The Bloch vector components are equal to the expectation values of the spin. If the tip of the Bloch vector reaches the surface of the sphere, this is a pure state. The state P_{Bloch} =0 (center of sphere) represents a completely depolarized spin.


mixture

Pure state


Entanglement and complementarity

Generic description of a two pathes quantum interference experiment involving a system A interacting with a « watching » system M which gets entangled with A :



$$\left|\Psi_{AM}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\psi_A(1)\right\rangle\left|\psi_M(1)\right\rangle + e^{i\varphi}\left|\psi_A(2)\right\rangle\left|\psi_M(2)\right\rangle\right)$$

The interference signal is calculated with the density operator of A, obtained by tracing over the states of M:

$$\rho_A = T_{rM}\left|\Psi_{A,M}\right\rangle\left\langle\Psi_{AM}\right| = \frac{1}{2}\left|\psi_A(1)\right\rangle\left\langle\psi_A(1)\right| + \frac{1}{2}\left|\psi_A(2)\right\rangle\left\langle\psi_A(2)\right|$$
$$+\frac{1}{2}\left|\psi_A(1)\right\rangle\left\langle\psi_A(2)\right|\left\langle\psi_M(2)\middle|\psi_M(1)\right\rangle e^{-i\varphi} + \frac{1}{2}\left|\psi_A(2)\right\rangle\left\langle\psi_A(1)\right|\left\langle\psi_M(1)\middle|\psi_M(2)\right\rangle e^{i\varphi}$$

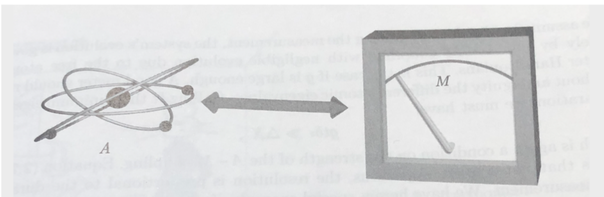
$$\Pi(x) = T_r\left(\left|x\right\rangle\left\langle x\right|\rho_A\right) = \left\langle x\right|\rho_A\left|x\right\rangle$$
$$= \frac{1}{2}\left|\left\langle x\middle|\psi_A(1)\right\rangle\right|^2 + \frac{1}{2}\left|\left\langle x\middle|\psi_A(2)\right\rangle\right|^2$$
$$+\mathrm{Re}\left(\left\langle x\middle|\psi_A(1)\right\rangle\left\langle\psi_A(2)\middle|x\right\rangle e^{-i\varphi}\times\left\langle\psi_M(2)\middle|\psi_M(1)\right\rangle\right)$$

The interference terms are in the cross products and the fringe contrast is equal to the modulus of the scalar product of the M system final states:

Fringe contrast: $\left|\left\langle\psi_M(2)\middle|\psi_M(1)\right\rangle\right|$

The more distinguishable the two M states are, the more information they store about the path of A and the smaller is the fringe contrast. The system M can be a measuring apparatus (like the moving slit) or a passive environment. In this case, the loss of quantum coherence is called decoherence. The principle of complementarity is directly related to the properties of entanglement (see below).

Entanglement and measurement



To measure an observable O_A of a microscopic system A, one has to couple it, via an amplifying scheme, to a macroscopic meter M whose state will be directly read out. The coupling A-M must correlate eigenstates with different eigenvalues of O_A to orthogonal states of M.

$$\left|\psi\right\rangle = \sum_i \alpha_i\left|a_i\right\rangle \quad ; \quad O_A\left|a_i\right\rangle = \varepsilon_i\left|a_i\right\rangle$$

$$\left|\psi\right\rangle_A\left|\xi_O\right\rangle_M \rightarrow \sum_i \alpha_i\left|a_i\right\rangle_A\left|\xi_i\right\rangle_M$$

The state $|\xi_0\rangle_M$ is the « neutral » state of the meter and the $|\xi_i\rangle_M$ are non-overlapping wave packets corresponding to different positions of the tip of the meter. Assuming that this tip moves along the Ox axis, the interaction can be described by the Hamiltonian:

$$H_{AM} = gO_AP_M$$

where P_M is the meter momentum along Ox and g a coupling parameter.

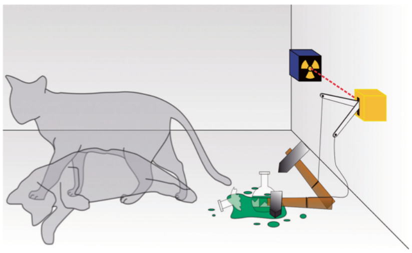
Recalling that the momentum operator is the generator of the meter translations, we then find easily:

$$U(t)\left|\psi\right\rangle_A\left|\xi_0\right\rangle_M = e^{-i\frac{gO_AP_M}{\hbar}t}\left|\psi\right\rangle_A\left|\xi_0\right\rangle_M$$
$$= \sum_i \alpha_i\left|a_i\right\rangle_A\left|\xi_0 + gt\varepsilon_i\right\rangle_M$$

M is entangled with A. Measuring the meter final position yields the result $\xi_0+gt\varepsilon_i$ with probability $|\alpha_i|^2$ and collapses A into $|a_i\rangle$. Calling $\Delta\xi$ the width of the meter wave packet and $\delta\varepsilon$ the smallest eigenvalue difference in the O_A spectrum, we must have:

$$gt\delta\varepsilon > \delta\xi$$

Schrödinger cat



Consider the measurement of the observable O_A of a two-level system (eigenstates $|0\rangle_A$ and $|1\rangle_A$) and call $|0\rangle_M$ and $|1\rangle_M$ the mirror states of the meter. Suppose we measure A in the superposition state:

$$|\psi\rangle_A = \frac{1}{\sqrt{2}}\left(|0\rangle_A + |1\rangle_A\right)$$

After interaction and before observing M, the A-M system turns in the entangled state:

$$|\psi\rangle_{AM} = \frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_M + |1\rangle_A|1\rangle_M\right)$$

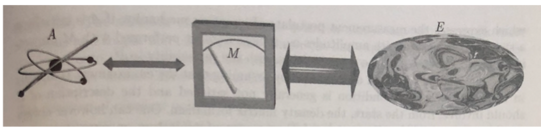
The meter being macroscopic, the measurement seems to prepare a superposition of states with different classical attributes: the tip of the meter points at the same time in different macrospically distinct directions: this looks like a **Schrödinger cat** (or like the moving slit in the Young thought experiment).

Moreover, the situation is ambiguous because $|\Psi\rangle_{AM}$ can be written as well as:

$$|\psi\rangle_{AM} = \frac{1}{2}\Big[\left(|0\rangle_A + |1\rangle_A\right)\left(|0\rangle_M + |1\rangle_M\right) + \left(|0\rangle_A - |1\rangle_A\right)\left(|0\rangle_M - |1\rangle_M\right)\Big]$$

So what is being measured here: the observable having eigenstates $|0\rangle_A$ and $|1\rangle_A$ or the one with eigentstates $|0\rangle_A+|1\rangle_A$ and $|0\rangle_A-|1\rangle_A$? These two observables do not commute and it is impossible that they could be measured indifferently by the same apparatus. The problem here is that we have forgotten the environment!

Entanglement, environment and decoherence



To solve the measurement paradox, consider that the meter M interacts with an environment E (gas molecules, thermal photons, internal degrees of freedom…). Quickly, information about M leaks into E, the M-E entanglement being realized, in a simple model, by the transformation:

$$|m\rangle_M|e\rangle_E \rightarrow |m\rangle_M|m\oplus e\rangle_E$$

where *m* and *e*, taking the values 0 and 1, represent the states of M and the states they are correlated to in E and $\oplus$ describes the addition modulo 2. Assuming that E is initially in state $|0\rangle_E$, the M-E interaction leaves $|0\rangle_M|0\rangle_E$ invariant and transforms $|1\rangle_M|0\rangle_E$ into $|1\rangle_M|1\rangle_E$.

After this interaction, the A-M system becomes tri-partite:

$$|\psi\rangle_{AME} = \frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_M|0\rangle_E + |1\rangle_A|1\rangle_M|1\rangle_E\right)$$

The information about the state of the A-M system being inscribed in E, the coherence between different M states is destroyed and the tri-partite A-M,E quantum state becomes an A-M density operator:

$$\rho_{AM} = \frac{1}{2}|0_A,0_M\rangle\langle 0_A,0_M| + \frac{1}{2}|1_A,1_M\rangle\langle 1_A,1_M|$$

This is again a manifestation of complementarity: E plays for M the role of a measuring device which transforms the state superpositions involving different meter states into a statistical mixture. The disappearance of the « cat's coherences » describes the phenomenon of **decoherence** which occurs very fast for macroscopic meters (back to this point later).

Note that in our model, the meter in states $|0\rangle_M$ and $|1\rangle_M$ couples with E *without entanglement*. This property defines the « ***pointer states*** » of the meter and suppresses the measurement ambiguity: a given apparatus measures the observable corresponding to its meter pointer states. It cannot be used to measure indifferently non-commuting observables.

Quantifying entanglement: The Schmidt expansion of a bipartite state

The partial density operator ρ_A of an A-B system is an hermitian operator in H_A, expressed in its diagonal basis as:

$$\rho_A = \sum_j \lambda_j |j_A\rangle\langle j_A|$$

The λ_j are positive numbers summing up to unity:

$$\lambda_j \geq 0 \quad \sum_j \lambda_j = 1$$

By disregarding the quantum coherence between A and B, we lost information about the entangled system S. Our knowledge of the state of A is reduced. Instead of knowing that it is, like $|\psi_S\rangle$, in a "pure" quantum state, we must describe it as a "statistical mixture of states" $|j_A\rangle$'s, with the distribution of probabilities given by the λ_j's. These probabilities are real numbers, not to be confused with the c-number amplitudes which interfere in the evolution of pure quantum states. Expanding over the $|j_A\rangle|\mu_B\rangle$ basis, we can always express the most general state of S as:

$$|\psi_S\rangle = \sum_{j,\mu} \alpha_{j\mu} |j_A\rangle \otimes |\mu_B\rangle = \sum_j |j_A\rangle \otimes |j_B\rangle$$

$$\text{with} \qquad |j_B\rangle = \sum_\mu \alpha_{j\mu} |\mu_B\rangle$$

The $|j_B\rangle$'s are, like the $|j_A\rangle$'s, orthogonal to each other as is easily shown:

$$\langle j_B | j'_B\rangle = \sum_\mu \alpha^*_{j\mu}\alpha_{j'\mu} = \langle j_A|\rho_A|j'_A\rangle = \lambda_j \delta_{jj'}$$

Finally, we can replace the $|j_B\rangle$'s by the states normalized to unity:

$$|j_B\rangle = |\tilde{j}_B\rangle / \sqrt{\lambda_j}$$

and we obtain the Schmidt expansion of the entangled state:

$$|\psi_S\rangle = \sum_j \sqrt{\lambda_j}\, |j_A\rangle \otimes |\tilde{j}_B\rangle$$

It associates to a basis $[|j_A\rangle]$ in H_A a «mirror» basis $[|j_B\rangle]$ in H_B. We can, without loss of generality, call A the part whose Hilbert space has a dimension n smaller or equal to that of the other part. If $n_B > n_A$, the "mirror basis" in H_B defines a subspace of B on which H_A is mirrored by the entanglement. The n-term Schmidt expansion is simpler and more informative than the expression of $|\Psi_S\rangle$ expanded over $n_A n_B$ terms.

Entropy of entanglement

The partial density operators ρ_A and ρ_B have identical spectra which describes the statistical common distribution of probabilities among the « mirror » states $|j_A\rangle$ and $|j_B\rangle$:

$$\rho_A = \sum_j \lambda_j |j_A\rangle\langle j_A| \qquad \rho_B = \sum_j \lambda_j |\tilde{j}_B\rangle\langle \tilde{j}_B|$$

To quantify the loss of information in the system parts, one defines the entropy of entanglement as:

$$S_A = S_B = -Tr\rho_A log_n(\rho_A) = -Tr\rho_B log_n(\rho_B)$$

$$= -\sum_j \lambda_j \log_n \lambda_j \qquad \left(\sum_j \lambda_j = 1\right)$$

This definition reminds the Boltzmann entropy of a gas of N molecules distributed among n cells, with a fraction $n_i = N_i/N$ particle in cell n° i:

$$S_{Boltzmann}/k_B = LogW = Log\frac{N!}{\Pi_i N_i!} \sim$$

$$NLogN - N - \sum_i \left(N_i LogN_i - N_i\right) =$$

$$NLogN - N\sum_i n_i(Logn_i + LogN) = -N\sum_i n_i Logn_i$$

As in the thermodynamical analog, the entropy of entanglement measures the disorder in the assignment of A (and B) to n quantum states, i.e. the loss of information about the two parts of the system when they are entangled. Log function in basis n is chosen so that maximum entropy is 1. Two limiting cases:

No entanglement (only one λ_j non zero)

$$S_A = S_B = 0 \quad \xrightarrow{one\ \lambda_j = 1} \quad |\psi_S\rangle = |j_A\rangle \otimes |\tilde{j}_B\rangle$$

All information is contained in parts A and B which are in separate pure states

Maximum entanglement (all λ_j equal):

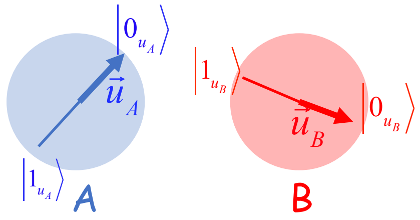
$$S_A = S_B = 1 \Rightarrow \quad |\psi_S\rangle = \frac{1}{\sqrt{n}}\sum_j |j_A\rangle \otimes |\tilde{j}_B\rangle \quad ;$$

$$\rho_A, \rho_B = \frac{1}{n} I :$$

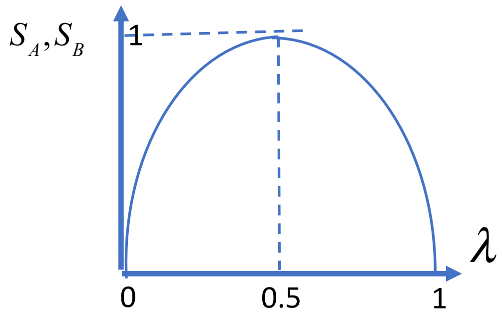
No information in A and B separately. All information in A/B correlations

Entanglement of spin-like two level systems

Most general entangled state of two spin-like particles



$$|\psi_S\rangle = \sqrt{\lambda}\,|0_{u_A}\rangle|0_{u_B}\rangle + \sqrt{1-\lambda}\,|1_{u_A}\rangle|1_{u_B}\rangle$$



Entropy of entanglement vs λ

Bell states: 4 mutually orthogonal maximally entangled spin-like states expressed in the basis of $\sigma_{zA}\sigma_{zB}$:

$$|\varphi_{BS}^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle \pm |1_A 1_B\rangle\right)$$

$$|\psi_{BS}^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|0_A 1_B\rangle \pm |1_A 0_B\rangle\right)$$

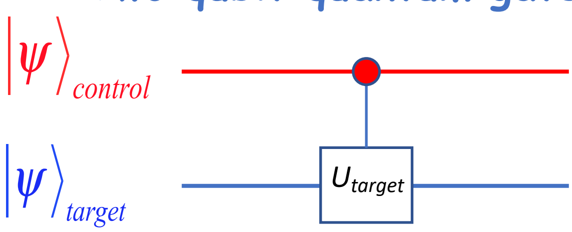
Three Bell states are symmetrical by particle exchange and have total spin 1

$$S^2|\varphi_{BS}^{\pm}\rangle = 2\hbar^2|\varphi_{BS}^{\pm}\rangle \quad ; \quad S^2|\psi_{BS}^{+}\rangle = 2\hbar^2|\psi_{BS}^{+}\rangle$$

One Bell state is antisymmetrical by exchange with S=0. It is invariant by all global rotations:

$$S_{x,y,z}|\psi_{BS}^{-}\rangle = 0 \qquad R_{\vec{u}}(\theta)|\psi_{BS}^{-}\rangle = |\psi_{BS}^{-}\rangle$$

Two qubit quantum gate



A two-qubit quantum gate realizes a *conditional dynamics*:

If control bit is in state |0> it remains in this state and the state of qubit B is unchanged:

$$|0_{control}\rangle|\psi_{target}\rangle \xrightarrow{\text{gate operation}} |0_{control}\rangle|\psi_{target}\rangle$$

If control bit is in state |1>, it remains in this state and the the state of qubit B is transformed by the unitary operation U_{target}:

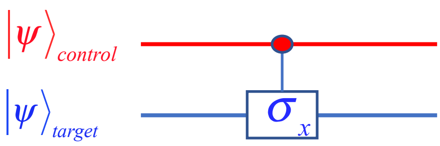
$$|1_{control}\rangle|\psi_{target}\rangle \xrightarrow{\text{gate operation}} |1_{control}\rangle \otimes U_{target}|\psi_{target}\rangle$$

By linearity:

$$\left(\alpha|0\rangle_c + \beta|1\rangle_c\right)\otimes|\psi\rangle_t \xrightarrow{\text{gate operation}}$$
$$\alpha|0\rangle_c|\psi\rangle_t + \beta|1\rangle_c \otimes U_{target}|\psi_{target}\rangle$$

A two-qubit quantum gate can entangle or disentangle two qubits

Two-qubit Quantum Gates: The control not gate



Truth table

$|0,0\rangle \to |0,0\rangle$

$|0,1\rangle \to |0,1\rangle$

$|1,0\rangle \to |1,1\rangle$

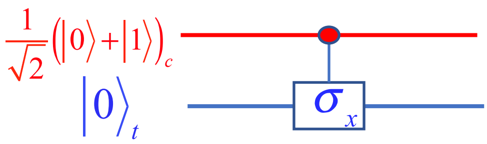
$|1,1\rangle \to |1,0\rangle$

Conditional dynamics: the control is not changed and its state determines the evolution of the target: the target flips if and only if the control is in |1>

The control-not gate realizes on the target state the *addition in base 2* of the two bits :

$$|a_c, a_t\rangle \to |a_c, a_c \oplus a_t\rangle \quad ; \quad (a_c, a_t = 0,1)$$

The control gate produces entanglement:



$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)_c |0\rangle_t \to \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle)$$

In reverse operation, the control gate disentangles two qubits:

$$\frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle) \to \frac{1}{\sqrt{2}}(|0,0\rangle + |1,0\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)_c |0\rangle_t$$

Two-qubit interaction realizing a control-not gate

Evolution of two qubit interacting via the Hamiltonian:

$$H = \varepsilon_0 (I - \sigma_z)_c \otimes (I - \sigma_x)_t$$

$$H|0,0\rangle = H|0,1\rangle = 0$$

$$H(|1,0\rangle + |1,1\rangle) = 0$$

$$H(|1,0\rangle - |1,1\rangle) = 2\varepsilon_0 (|1,0\rangle - |1,1\rangle)$$

H spectrum: 3 eigenstates with 0 eigenvalue, 1 eigenstate with eigenvalue $2\varepsilon_0$

$$|0,0\rangle \to |0,0\rangle \qquad\qquad |0,1\rangle \to |0,1\rangle$$

$$|1,0\rangle \xrightarrow[U(t)]{} \frac{1}{2}(|1,0\rangle + |1,1\rangle) + \frac{e^{-2i\varepsilon_0 t/\hbar}}{2}(|1,0\rangle - |1,1\rangle)$$

$$= \frac{1}{2}\left(1 + e^{-2i\varepsilon_0 t/\hbar}\right)|1,0\rangle + \frac{1}{2}\left(1 - e^{-2i\varepsilon_0 t/\hbar}\right)|1,1\rangle$$

$$|1,1\rangle \xrightarrow[U(t)]{} \frac{1}{2}(|1,0\rangle + |1,1\rangle) - \frac{e^{-2i\varepsilon_0 t/\hbar}}{2}(|1,0\rangle - |1,1\rangle)$$

$$= \frac{1}{2}\left(1 - e^{-2i\varepsilon_0 t/\hbar}\right)|1,0\rangle + \frac{1}{2}\left(1 + e^{-2i\varepsilon_0 t/\hbar}\right)|1,1\rangle$$

$$2\varepsilon_0 t/\hbar = \pi \quad \Longrightarrow \quad \begin{aligned} |1,0\rangle &\to |1,1\rangle \\ |1,1\rangle &\to |1,0\rangle \end{aligned}$$

See later how to realize in practice this Hamiltonian

Entanglement and non-locality: the EPR paper

Einstein, Podolski and Rosen (EPR) analyzed measurements on an entangled system of 2 particles performed at the same time by two observers (named Alice and Bob in the language of quantum information) located at an arbitrary large distance from each other. Only one spatial dimension is considered.The particles, prepared in the past in an entangled state through a non-specified process, do not interact at the time of the measurement.

The entangled state at the time of measurement (t=0) expanded on the basis of the particles momenta exhibits maximum entanglement:

$$|\psi(0)\rangle = \int dp e^{-ipr_0/\hbar} |p\rangle_A \otimes |-p\rangle_B$$

In the position basis, the same state writes:

$$\psi(x_A, x_B; t=0) = \langle x_A, x_B | \psi(0)\rangle =$$

$$\int dp e^{ip(x_A - x_B - r_0)/\hbar} = \delta(x_A - x_B - r_0)$$

where r_0 is a free parameter fixed during the preparation of the state, for instance equal to the distance separating Alice from Bob.

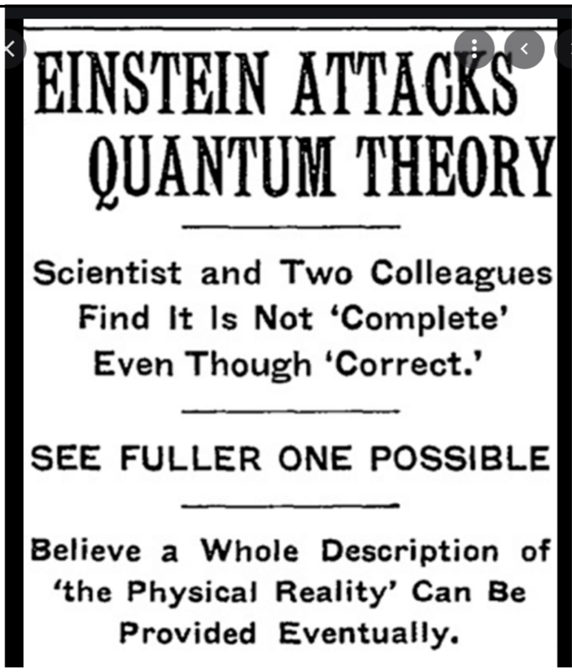
It results from these expressions that the measurements of momenta and positions by Alice and Bob are perfectly correlated, with eigenvalues satisfyng the conditions:

$$p_A + p_B = 0 \quad ; \quad x_A - x_B = r_0$$

These formulas illustrate the essential feature of entanglement: the position and momentum of each part is completely undetermined, while their correlations are maximum. The operators X_A-X_B and P_A+P_B can be simultaneously measured because these two observables of the global system commute:

$$[X_A - X_B, P_A + P_B] = 0$$

New York Times,
May 4th 1935



EINSTEIN ATTACKS QUANTUM THEORY

Scientist and Two Colleagues Find It Is Not 'Complete' Even Though 'Correct.'

SEE FULLER ONE POSSIBLE

Believe a Whole Description of 'the Physical Reality' Can Be Provided Eventually.

Einstein' argument about incompleteness of quantum physics

Alice can determine the position or velocity of the particle she detects without having to touch it or to interact with it in any manner. She can ask Bob to measure X or P and to tell her the result. Due to the perfect correlations of the entangled state, she then knows with certainty the values of X or P for her particle.

The situation looks similar to a classical game: A third player paints a blue and a red spot on two balls sealed in two envelopes and sends one to Alice, the other to Bob. Alice does not need to open her envelope to find out what color she received. She just needs Bob to tell her his result. The game can also be modified to simulate classicaly the non-commutation of X and P. The balls might be small balloons of slightly different sizes. Opening the box and exposing them to light to see the color makes them explode, preventing to find out their size.

Probing the size must be done by touching the balls in the dark, but this contact erases the paint and the information about the color of the ball. For each envelope she receives, Alice can, without opening it, know for sure either the color or the size of the ball by asking Bob to check for her. Einstein concluded that the quantities X or P which Alice could obtain without interacting with her particle. must have been, like the color or the volume of the ball in the classical experiment, already existing prior to Bob's measurement. They must have been « *elements of reality* » created in the process which prepared the entangled state. This preparation must have involved *hidden variables*, not expressed in the wave function, meaning that the description of the physical world given by quantum physics must be incomplete.

Spin version of the EPR experiment

The EPR thought experiment with continuous variables presents many difficulties. The preparation of the entangled state, which is fully delocalized in position as well as in momentum space is not specified which makes the description of an experiment difficult (what is the apparatus used to prepare the state?). Moreover, the maximum entanglement in X occurs only at t = 0. EPR did not consider the time dependence of the state, which does not conserve the perfect spatial correlations because X_A-X_B does not commute with the global Hamiltonian of the free particles:

$$\left[X_A - X_B, \frac{P_A^2}{2m} + \frac{P_B^2}{2m}\right] = \frac{i\hbar}{m}\left(P_A - P_B\right)$$

The EPR wave function becomes at time t:

$$\psi(x_A, x_B; t) = \left\langle x_A, x_B \middle| \psi(t) \right\rangle = \int dp e^{ip(x_A - x_B - r_0)/\hbar} e^{-ip^2 t/m\hbar}$$

and, due to the quadratic term in the p-exponential,the full correlation between X_A and X_B is lost for $t \neq 0$

A first improvement in the EPR problem was made by Bohm in the 1950's who showed that the Einstein arguments could be repeated by considering a system of two spin systems on which Alice and Bob measure non commuting spin components. The situation becomes simpler because the continuous integrals become discrete sums and the observables of the entangled system are now time independent.
A second critical step was achieved by J.Bell in 1964, who proposed a feasible thought experiment in order to check whether the assumption of EPR that quantum mechanics is incomplete can be validated or contradicted. This is the so called « *Bell's inequality test* » which is presented on the next slides, along with the first crucial experimental implementation.

Non-locality in the spin version of the EPR experiment

Assume that Alice and Bob share a pair of particles prepared in the Bell's state:

$$\left|\varphi_{BS}^{+}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0_A,0_B\right\rangle + \left|1_A,1_B\right\rangle\right)$$

If A and B measure their particles in the same Oz direction, they always find the same result, which is randomly +1 if the system is projected on $|0_A0_A\rangle$, -1 if it collapses in $|1_A1_A\rangle$.
Let us show that the same perfect correlation holds if the measurements are performed along any arbitrary direction **u** in the xOz plane.

The tensor product of the Pauli operators in the direction making an angle θ with Oz is:

$$\sigma_u^A\sigma_u^B = \cos^2\theta\sigma_z^A\sigma_z^B + \sin^2\theta\sigma_x^A\sigma_x^B + \sin\theta\cos\theta\left(\sigma_x^A\sigma_z^B + \sigma_z^A\sigma_x^B\right)$$

It is easy to check that the action of the first two terms on $|\varphi^+_{BS}\rangle$ leaves this state unchanged for any θ, while the last term gives 0:

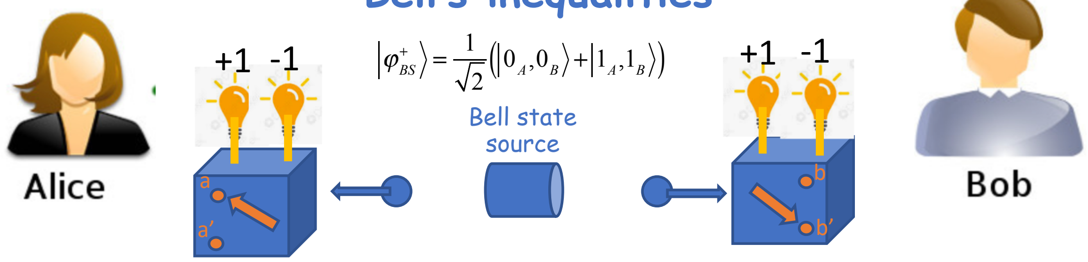
$$\sigma_u^A\sigma_u^B\left|\psi_{BS}^{+}\right\rangle = \left|\psi_{BS}^{+}\right\rangle$$

The $\sigma_u^A\sigma_u^B$ operator is diagonal in the spin basis pointing along u. Its degenerate eigenvalues are +1 with eigenstates $|0,0\rangle_u$ and $|1,1\rangle_u$ and -1 with eigenstates $|0,1\rangle_u$ and $|1,0\rangle_u$
The fact that $\sigma_u^A\sigma_u^B$ leaves $|\varphi^+_{BS}\rangle$ invariant means that this state belong to the subspace with eigenvalue +1 of this operator. In other words, measuring the two particles of the pair along any arbitrary direction u must yield a perfect correlations: both results must be either +1 or -1.

Einstein's argument then applies: Alice can know the value of any component of her particle's spin in the xOz plane without touching her particle, by merely asking Bob to measure his particle and tell her the result. One is thus forced to admit either the « *spooky action at a distance* » implied by non-locality, or the existence of hidden variables. The discussion made here with one of the 4 Bell states could be repeated with the other 3.

Bell's inequalities

$$\left|\varphi_{BS}^{+}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0_A,0_B\right\rangle + \left|1_A,1_B\right\rangle\right)$$



Alice and Bob share pairs of spins prepared in the Bell state $|\varphi^+_{BS}\rangle$. For each pair, A measures the spin of her particle along one of two directions $\mathbf{u}_a$ or $\mathbf{u}_{a'}$ while Bob measures his particle's spin along either u_b or $u_{b'}$. The four possible results are $\varepsilon_a,\varepsilon_{a'},\varepsilon_b,\varepsilon_{b'}$ equal to +1 or -1. From a large set of pairs analyzed in this way, Alice and Bob get the averages of the four correlations $\varepsilon_a\varepsilon_b$, $\varepsilon_a\varepsilon_{b'}$, $\varepsilon_{a'}\varepsilon_b$ and $\varepsilon_{a'}\varepsilon_{b'}$.

If the ε's are « elements of reality » as assumed by Einstein, all must exist for each pair, even if only two can be actually measured due to the non-commutation of the Pauli operators. The ε satisfy obviously the equality:

$$\sum_B = (\varepsilon_a - \varepsilon_{a'})\varepsilon_b + (\varepsilon_a + \varepsilon_{a'})\varepsilon_{b'} = \pm 2$$

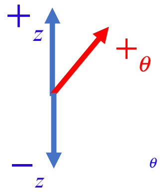
And the average of this sum over a large number of realizations (each one contributing to one out of the four possible terms) is:

$$\left\langle \sum_B \right\rangle = \left\langle\varepsilon_a\varepsilon_b\right\rangle - \left\langle\varepsilon_{a'}\varepsilon_b\right\rangle + \left\langle\varepsilon_a\varepsilon_{b'}\right\rangle + \left\langle\varepsilon_{a'}\varepsilon_{b'}\right\rangle$$ It is obviously bounded by -2 and +2: $$-2 \le \sum_B \le +2$$

This Bell inequality is violated by quantum physics for some spin polarizations

Quantum mechanical calculation of the Bell sum

According to quantum physics, the mean values of the ε products are given by the expectation values of products of Pauli operators. Measuring a spin first in one direction, then at an angle θ with that direction yields a mean value for the second measurement equal to $\cos^2(\theta/2) - \sin^2(\theta/2) = \cos\theta$. The correlation between a measurement by Alice and one by Bob made with spin directions making an angle θ_{ab} is thus $\langle\sigma_a\sigma_b\rangle = \cos\theta_{ab}$ and we get for the Bell sum:

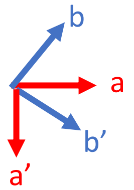


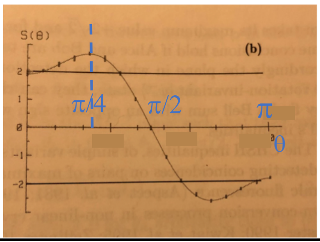
First measurement projects spin in direction $+\theta$. The mean value of spin along Oz is then:

$$_\theta\langle +|\sigma_z|+\rangle_\theta = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) = \cos\theta$$

$$\sum_B = \cos\theta_{ab} - \cos\theta_{a'b} + \cos\theta_{ab'} + \cos\theta_{a'b'}$$

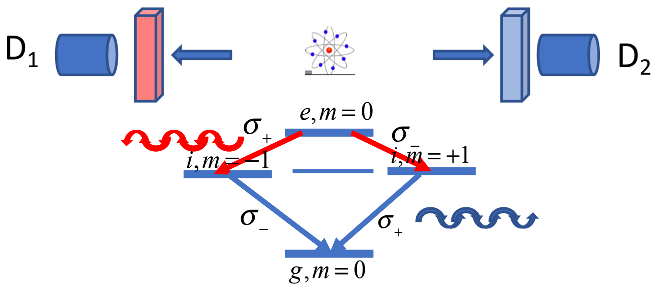
Make the angle choice: $\theta_{ab} = \theta_{b'a} = \theta_{a'b'} = \theta \quad ; \quad \theta_{a'b} = \theta_{a'b'} + \theta_{b'a} + \theta_{ab} = 3\theta$

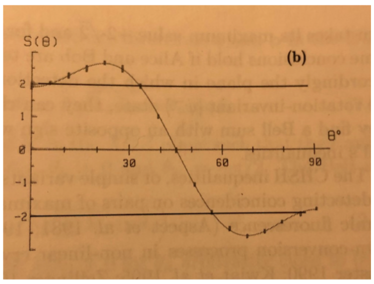




The Bell sum 3cosθ-cos3θ plotted versus θ exhibits regions of θ values violating Bell's inequality. Maximum violation occurs for $\theta=\pi/4$ and $3\pi/4$($+2\sqrt{2}$ *and* $-2\sqrt{2}$ respectively). Points are experimental (see next slide).

Aspect's experiment

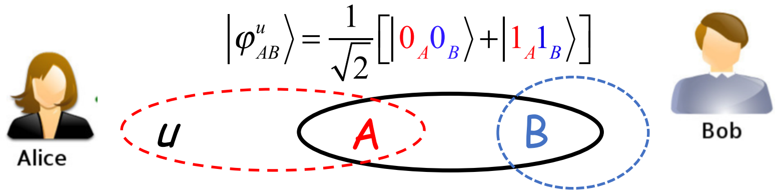




The Bell's inequality test has been realized by Aspect in 1982 with pairs of blue/red photons emitted by Calcium atoms in a radiating cascade. The photons were detected with four different polarizations. The polarizers were randomly switched in a time interval shorter than the one it takes for light to travel from one detector to the other, to preclude any possible information transfer between them. Correlations similar to the ones discussed in the spin model are collected (*remark that two orthogonal states of linear light polarization correspond to polarizers oriented at 90°, while for the spins, state othogonality means that the spins make an angle of 180°).* When translated in spin language, the variation of the Bell sum as a function of θ agrees with the prediction of quantum mechanics. The Bell's sum maximum or minimum $\pm 2\sqrt{2}$ are well outside the bounds of hidden variable predicted values (see figure on left where points are experimental and the curve represents the 3cosθ-cos3θ function)

Quantum teleportation

$$\left|\varphi_{AB}^{u}\right\rangle = \frac{1}{\sqrt{2}}\left[\left|0_A 0_B\right\rangle + \left|1_A 1_B\right\rangle\right]$$



1. Alice and Bob separated by a large distance share a pair of entangled qubits A,B

2. Alice is given a qubit u prepared in a state $|\Psi_u\rangle$ unknown by her and by Bob and her task is to send a copy of this state to Bob

$$\left|\Psi_u\right\rangle = \alpha\left|0_u\right\rangle + \beta\left|1_u\right\rangle$$

3. Alice performs on her side a joint measurement on the system u+A and finds one out of four possible results. This measurement induces the collapse of the B qubit of Bob in one state out of four possible states.

4. Alice communicates her result to Bob in the form of a two bit sequence and Bob, depending on this classical information, performs a unitary operation on qubit B which brings it in state $|\Psi_u\rangle$.

Quantum Teleportation: Alice's measurement

Let us expand the initial state of the three qubit system (u,A,B) on the tensor product basis |a,b,c> (a,b,c=0,1):

$$\sqrt{2}\left|\Psi_u\right\rangle\left|\phi_{AB}^{u}\right\rangle = \left[\alpha\left|0_u\right\rangle + \beta\left|1_u\right\rangle\right]\left[\left|0_A 0_B\right\rangle + \left|1_A 1_B\right\rangle\right]$$
$$= \alpha\left(\left|0_u 0_A\right\rangle\left|0_B\right\rangle + \alpha\left|0_u 1_A\right\rangle\left|1_B\right\rangle\right) + \beta\left(\left|1_u 0_A\right\rangle\left|0_B\right\rangle + \left|1_u 1_A\right\rangle\left|1_B\right\rangle\right)$$

Then change basis to expand the three qubit state on the tensor product basis of the Bell's states of the u,A couple of qubits possessed by Alice by the qubit B under the control of Bob:

(u,A) Bell states:

$$\left|\varphi_{u,A}^{\pm}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0_u 0_A\right\rangle \pm \left|1_u 1_A\right\rangle\right)$$
$$\left|\psi_{u,A}^{\pm}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0_u 1_A\right\rangle \pm \left|1_u 0_A\right\rangle\right)$$

Change of basis for (u,A) system

$$\left|0_u 0_A\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\phi_{uA}^{+}\right\rangle + \left|\phi_{uA}^{-}\right\rangle\right) \quad ; \quad \left|1_u 1_A\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\phi_{uA}^{+}\right\rangle - \left|\phi_{uA}^{-}\right\rangle\right)$$
$$\left|0_u 1_A\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\psi_{uA}^{+}\right\rangle + \left|\psi_{uA}^{-}\right\rangle\right) \quad ; \quad \left|1_u 0_A\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\psi_{uA}^{+}\right\rangle - \left|\psi_{uA}^{-}\right\rangle\right)$$

and we rewrite the initial state as:

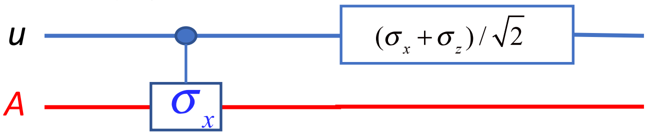
$$2\left|\Psi_u\right\rangle\left|\phi_{AB}^{u}\right\rangle = \left|\phi_{uA}^{+}\right\rangle\left[\alpha\left|0_B\right\rangle + \beta\left|1_B\right\rangle\right] + \left|\phi_{uA}^{-}\right\rangle\left[\alpha\left|0_B\right\rangle - \beta\left|1_B\right\rangle\right]$$
$$+\left|\psi_{uA}^{+}\right\rangle\left[\alpha\left|1_B\right\rangle + \beta\left|0_B\right\rangle\right] + \left|\psi_{uA}^{-}\right\rangle\left[\alpha\left|1_B\right\rangle - \beta\left|0_B\right\rangle\right]$$

Alice thus needs to measure the observable of the (A,u) system admitting Bell's states as eigenstates in order to collapse Bob's qubit in one out four superposition states

Quantum Teleportation: Alice's measurement (continued)

$$2|\Psi_u\rangle|\phi^u_{AB}\rangle = |\phi^+_{uA}\rangle\left[\alpha|0_B\rangle+\beta|1_B\rangle\right] + |\phi^-_{uA}\rangle\left[\alpha|0_B\rangle-\beta|1_B\rangle\right]$$
$$+|\psi^+_{uA}\rangle\left[\alpha|1_B\rangle+\beta|0_B\rangle\right] + |\psi^-_{uA}\rangle\left[\alpha|1_B\rangle-\beta|0_B\rangle\right]$$

Alice realizes a control-Not gate with qubit u as control and A as target, followed by a one qubit unitary operation on qubit u:

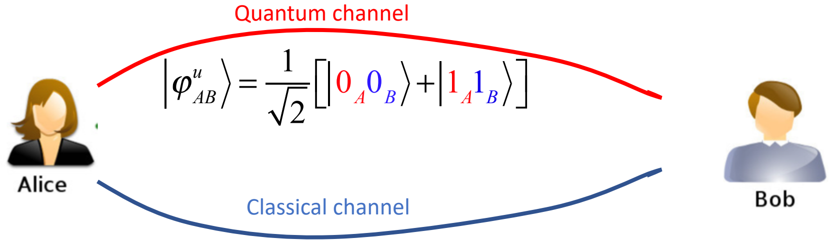


$$|\varphi^\pm_{u,A}\rangle = \frac{1}{\sqrt{2}}\left(|0_u 0_A\rangle \pm |1_u 1_A\rangle\right) \xrightarrow{\text{control not}} \frac{1}{\sqrt{2}}\left(|0_u\rangle \pm |1_u\rangle\right)\otimes|0_A\rangle \xrightarrow{(\sigma_x+\sigma_z)/\sqrt{2}} \begin{cases} \xrightarrow{\varphi^+_{uA}} |0_u\rangle|0_A\rangle \\ \xrightarrow{\varphi^-_{uA}} |1_u\rangle|0_A\rangle \end{cases}$$

$$|\psi^\pm_{u,A}\rangle = \frac{1}{\sqrt{2}}\left(|0_u 1_A\rangle \pm |1_u 0_A\rangle\right) \xrightarrow{\text{control not}} \frac{1}{\sqrt{2}}\left(|0_u\rangle \pm |1_u\rangle\right)\otimes|1_A\rangle \xrightarrow{(\sigma_x+\sigma_z)/\sqrt{2}} \begin{cases} \xrightarrow{\psi^+_{uA}} |0_u\rangle|1_A\rangle \\ \xrightarrow{\psi^-_{uA}} |1_u\rangle|1_A\rangle \end{cases}$$

Alice gets a two-bit information indicating on which Bell state the (u,A) system has collapsed and she sends by a classical channel the values of these bits to Bob.

Quantum Teleportation: Bob's action



$$0,0 \text{ or } 1,0 \text{ or } 0,1 \text{ or } 1,1$$

$$2|\Psi_u\rangle|\phi^u_{AB}\rangle = |\phi^+_{uA}\rangle\left[\alpha|0_B\rangle+\beta|1_B\rangle\right] + |\phi^-_{uA}\rangle\left[\alpha|0_B\rangle-\beta|1_B\rangle\right]$$
$$+|\psi^+_{uA}\rangle\left[\alpha|1_B\rangle+\beta|0_B\rangle\right] + |\psi^-_{uA}\rangle\left[\alpha|1_B\rangle-\beta|0_B\rangle\right]$$

Depending on the classical information received from Alice, Bob applies an adapted unitary transformation U_B on qubit B:

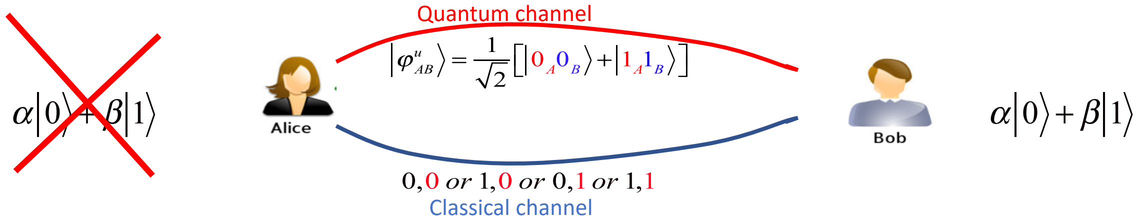
$$0,0 \rightarrow |\phi^+_{uA}\rangle \quad U_B = I \qquad 0,1 \rightarrow |\psi^+_{uA}\rangle \quad U_B = \sigma_x$$
$$1,0 \rightarrow |\phi^-_{uA}\rangle \quad U_B = \sigma_z \qquad 0,1 \rightarrow |\psi^-_{uA}\rangle \quad U_B = i\sigma_y$$

In all cases, Qubit B ends in state

$$\alpha|0\rangle + \beta|1\rangle$$

which completes teleportation

The main features of teleportation



Neither Alice nor Bob need to know the state which has been teleported

Teleportation destroys the state on Alice's side (otherwise, it would be a form of cloning)

The transfer of quantum information from Alice to Bob does not violate causality since it requires a classical communication channel which cannot be superluminal

A two-bit transfer is enough to communicate an information about two probability amplitudes (α,β) which would require many more bits to be communicated by completely classical means

Teleportation is an essential procedure in quantum information protocols

Practical implementation of teleportation will be described in later lecture

Concluding remarks

In lectures 4 to 7, I have introduced the principles and main features of quantum physics, a science which has started by general considerations about some mysterious properties of light, and has disclosed to us the microscopic world of atoms and photons.

I will describe in the next lectures several applications in the domains of electromagnetism and optics which have been made possible by the knowledge of the quantum laws: magnetic resonance, optical pumping, the maser and the laser, the atomic clocks and the GPS are among the inventions which have changed our lives during the second part of the twentieth century.These technological advances have also made possible fundamental experiments in basic science: very high resolution spectroscopy of atoms and molecules, physics of ultra-short and ultra intense light pulses, investigation of extremely low-temperature phenomena…

Describing the progresses in the fundamental and applied aspects of the science of light during the last century illustrates the symbiotic link between « blue sky » science motivated by curiosity and « useful science » aiming at solving practical or societal problems. One cannot develop and thrive without the other.