"Enrico Fermi" Chair
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A History of the Science of Light

From Galileo's telescope to the laser and the quantum information technologies

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Lecture 6. March 3rd 2022

Lecture 6: Angular momentum, spin and statistics

This lecture is devoted to the quantum description of the angular momentum (AM) and of the spin of quantum particles (electrons and nuclei). Classically the angular momentum measures the amount of rotation stored in a body, constituting the rotational analog of the linear momentum. In quantum physics the orbital AM associated to the spatial motion of a particle is the cross product $\vec{L}$ of its position and momentum operators. The x,y and z components of the AM do not commute between themselves but they do commute with the square L^2 of the AM which has a discrete spectrum of eigenvalues $l(l+1)\hbar^2$, with l=0,1,2…. For each l value, the L_i components (i=x,y,z) have a discrete spectrum varying by steps of $\hbar$ from $-l\hbar$ to $+l\hbar$. The component of the AM along a spatial axis is the generator of the rotations around this axis.

In addition to the orbital AM, quantum particles have a spin, of relativistic origin, which can be viewed as related to an intrinsic dynamical rotation of the particle around itself. The spin has like the orbital AM a discrete spectrum. The l *value* of the electron spin is equal to $\frac{1}{2}$ and its component along any given direction takes the values $\pm\hbar/2$ (spin $\frac{1}{2}$). The quantum states associated to the electron spin evolves in a two-dimension Hilbert space whose operators can be expressed as linear combinations of the identity operator and three Pauli operators proportional to the spin components along three orthogonal spatial directions. We describe the properties of these operators and of their eigenstates, using the convenient representation of the Bloch sphere.This representation is useful to describe any system evolving in a two-dimension Hilbert space like the qubits in quantum information. We will extensively use this representation in subsequent lectures.

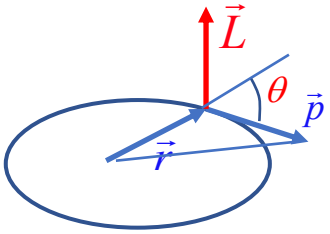
The value of the spin of particles is related to the statistics they obey to (bosons or fermions). We analyse this connexion by discussing the symmetrization postulate of quantum physics and we describe how it accounts for some important properties of bosons and fermions.

Finally, we briefly present the relativistic Dirac equation, which gave a theoretical foundation to the existence of the electron spin, predicted the positron and introduced the concepts of quantum field theory.

Orbital angular momentum

Angular Momentum in classical physics: astronomical example

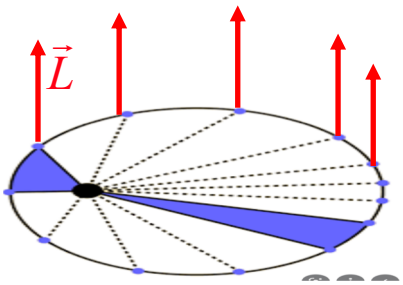
$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$ $\quad |\vec{L}| = r \cdot p \sin\theta$



In classical non-relativistic mechanics, the angular momentum of a body orbiting in a spherically symmetrical potential is invariant:

The second Kepler law states that the area *A* swept in a unit time by the radial vector joining the sun to a planet is a constant

$$r v \sin\theta dt = \frac{|L|\,dt}{m} = A dt$$



In quantum mechanics, the angular momentum is a vector operator:

$$L_x = YP_z - ZP_y$$
$$L_y = ZP_x - XP_z$$
$$L_z = XP_y - YP_x$$

Commutation rules: $[L_x, L_y] = i\hbar L_z ; [L_y, L_z] = i\hbar L_x ; [L_z, L_x] = i\hbar L_y$

$$L^2 = L_x^2 + L_y^2 + L_z^2 \qquad [L_i, L^2] = 0 \quad (i = x, y, z)$$

Spectrum of orbital angular momentum

$$L^2 |l,m\rangle = l(l+1)\hbar^2 |l,m\rangle \quad ; \quad l = 0,1,2.....$$
$$L_z |l,m\rangle = m\hbar |l,m\rangle \quad ; \quad -l \le m \le l$$

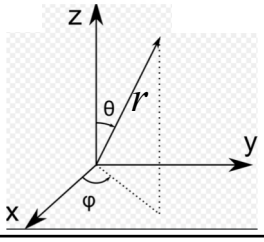
Classically, *m* corresponds to the discrete projections of L on Oz axis (spatial quantization)

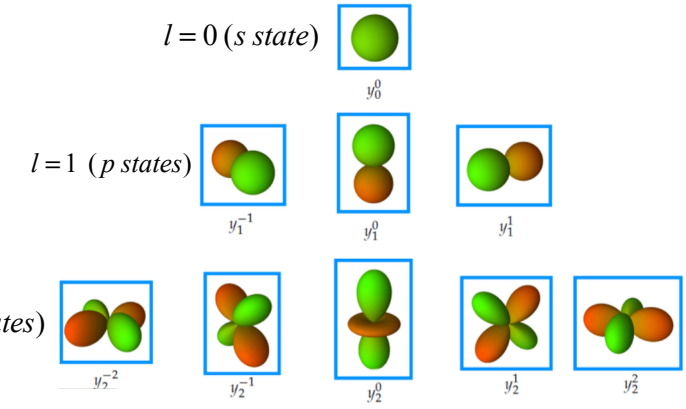
Since L^2 and L_z commute, there is a common basis of eigenstates for these two operators. An *l* state has a *2l+1* degeneracy (here *l=3*)

Eigenstates of angular momentum (spherical coordinate representation):

$$\langle r,\theta,\varphi | \psi_l^m \rangle = R(r) Y_l^m(\theta,\varphi)$$

Arbitrary function of r — Spherical harmonics





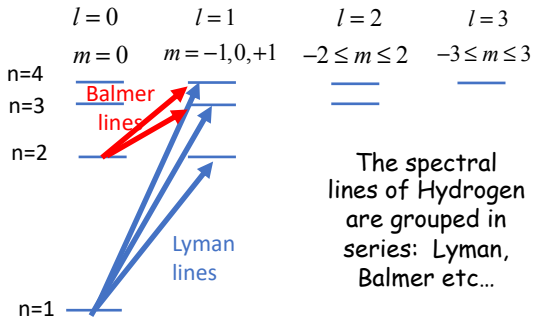
The hydrogen spectrum in non-relativistic quantum mechanics

$$H=\frac{P^2}{2m}-\frac{e^2}{r} \qquad \left[H,L_i\right]=\left[H,L^2\right]=0$$

There is a common basis of eigenstates of H, L^2 and L_z:

$$H\left|n,l,m\right\rangle = E_n\left|n,l,m\right\rangle \quad ; \quad E_n=-\frac{R_y}{n^2}$$
$$L^2\left|n,l,m\right\rangle = l(l+1)\hbar^2\left|n,l,m\right\rangle \quad ; \quad l=0,1,2....n-1$$
$$L_z\left|n,l,m\right\rangle = m\hbar\left|n,l,m\right\rangle \quad : \quad -l\le m\le l$$

$$\left\langle r,\theta,\varphi\middle|n,l,m\right\rangle = R_n(r)Y_l^m(\theta,\varphi)$$



The spectral lines of Hydrogen are grouped in series: Lyman, Balmer etc…

Orbital degeneracy of n^{th} manifold:

$$d_n=1+3+....+2n-1=n^2$$

Angular momentum and rotations

The operator: $R_Z(\alpha)=\exp(-i\alpha L_z/\hbar)$

rotates the position eigenstate of a particle in real space by the angle α around Oz:

$$R_Z(\alpha)\left|r,\theta,\varphi\right\rangle = \left|r,\theta,\varphi+\alpha\right\rangle$$

We start by demonstrating the identity in the spherical coordinate representation:

$$L_z=(\hbar/i)\partial/\partial\varphi$$

This relation results from the definition of spherical coordinates:

$$x=r\sin\theta\cos\varphi\,;\ y=r\sin\theta\sin\varphi\,;\ z=r\cos\theta$$
$$\frac{\partial}{\partial\varphi}=\frac{\partial}{\partial x}\frac{\partial x}{\partial\varphi}+\frac{\partial}{\partial y}\frac{\partial y}{\partial\varphi}=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}=\frac{i}{\hbar}L_z$$

We then express that the *l,m* spherical harmonic Y_l^m is an eigenstate of L_z with eigenvalue m:

$$L_zY_l^m=\frac{\hbar}{i}\frac{\partial}{\partial\varphi}Y_l^m=m\hbar Y_l^m$$

which shows that the φ dependance of Y_l^m is $e^{im\varphi}$:

$$Y_l^m(\theta,\varphi)=e^{im\varphi}f(r,\theta)$$

From the definition of $R_z(\alpha)$ we then get :

$$R_Z(\alpha)Y_l^m=e^{im(\varphi-\alpha)}f(r,\theta)$$
$$\rightarrow R_Z(\alpha)Y_l^m(r,\theta,\varphi)=Y_l^m(r,\theta,\varphi-\alpha)$$

which also writes:

$$\left\langle r,\theta,\varphi\middle|R_Z(\alpha)\middle|l,m\right\rangle=\left\langle r,\theta,\varphi-\alpha\middle|lm\right\rangle$$

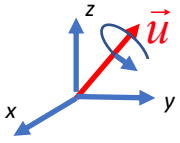
and, by hermitian conjugation:

$$\left\langle l,m\middle|R_z^\dagger(\alpha)\middle|r,\theta,\varphi\right\rangle=\left\langle lm\middle|r,\theta,\varphi-\alpha\right\rangle$$

and finally noting that $R_z^\dagger(\alpha)=R_z(-\alpha)$ and changing α in $-\alpha$ we get the announced result.

More generally, the operator associated to the rotation of angle α around the direction defined by vector **u** is:

$$R_{\vec u}(\alpha)=e^{-i\alpha\vec u.\vec L/\hbar}$$



Recapitulation: Unitary operators describing translations in position and momentum, rotations and time evolution

Translation in space:

$$T_{\vec{r}_0} = \exp(-i\vec{r}_0\vec{P}/\hbar) \quad ; \quad T_{\vec{r}_0}\left|\vec{r}\right\rangle = \left|\vec{r}+\vec{r}_0\right\rangle \quad ; \quad T_{\vec{r}_0}^{\dagger} = T_{-\vec{r}_0} = T_{\vec{r}_0}^{-1}$$

Momentum translation:

$$T_{\vec{p}_0} = \exp(i\vec{p}_0\vec{R}/\hbar) \quad ; \quad T_{\vec{p}_0}\left|\vec{p}\right\rangle = \left|\vec{p}+\vec{p}_0\right\rangle \quad ; \quad T_{\vec{p}_0}^{\dagger} = T_{-\vec{p}_0} = T_{\vec{p}_0}^{-1}$$

Rotation in space

Rotated Position

$$R_{\vec{u},\alpha} = \exp(-i\alpha\vec{u}.\vec{L}) \; ; \quad R_{\vec{u},\alpha}\left|\vec{r}\right\rangle = \left|\mathrm{R}(\vec{r})\right\rangle \quad ; \quad R^{\dagger}_{\vec{u},\alpha} = R_{\vec{u},-\alpha} = R^{-1}_{\vec{u},\alpha}$$

Time evolution:

$$U(\Delta t) = \exp(-i\Delta t H/\hbar) \quad ; \quad U(\Delta t)\left|\psi(t)\right\rangle = \left|\psi(t+\Delta t)\right\rangle \quad ; \quad U^{\dagger}(\Delta t) = U(-\Delta t) = U^{-1}(\Delta t)$$

These four operators are unitary: their hermitian conjugate realizes the inverse operation

Spin angular momentum

In addition to the orbital momentum described above, elementary particles have an intrinsic angular momentum or spin. It is the case of the electron and of the proton. Their spins can classically be viewed as the angular momentum associated to the spinning of the particles around an axis passing through them (in analogy with the diurnal rotation of the Earth and Sun). The spin is however a quantum-relativistic concept and the classical analogy must be used with caution. The spin S of the electron and of the proton is a vector angular momentum observable with 3 components S_i (i=x,y,z):

$$\vec{S} \quad (S_x, S_y, S_z)$$

It obeys the same commutation relations than the orbital AM:

$$\left[S_i, S_j\right] = i\hbar\varepsilon_{ijk}S_k \; ; \quad \left[S_{i,}, S^2\right] = 0$$

$$(i,j,k = x,y,z; \varepsilon_{ijk} = \pm \; for \; even/odd$$

$$permutations \; of \; indices \; x,y,z.$$

Classical description of Hydrogen atom with the intrinsic spins of electron and proton

S^2 and S_z commuting, these two observable share a common eigenbasis.

$$\left|S, m_s\right\rangle \quad (S = \frac{1}{2} \; ; \; m_s = \pm\frac{1}{2})$$

with the eigenvalues equations:

$$S_z\left|S, m_S\right\rangle = m_S\left|S, m_S\right\rangle$$

$$S^2\left|S, m_s\right\rangle = S(S+1)\hbar^2\left|S, m_s\right\rangle = \frac{3\hbar^2}{4}\left|S, m_s\right\rangle$$

Contrary to the orbital AM which is an integer, the electron spin has a half-integer value. It evolves in a Hilbert space of dimension d =2S+1=2. On next slide, we introduce a convenient formalism to describe the states and the operators in this Hilbert space.

Spin ½ as a generic model of a two-level quantum system

A spin ½ evolves in a two dimension Hilbert space which can be represented in the basis [$|+\rangle_z$,$|-\rangle_z$] of the S_z eigenstates (eigenvalues$\pm\hbar/2$; so called "spin up" and "spin down" states). It is a generic model for a quantum system whose all states save two can be ignored in a specific situation (atom evolving between a ground and an excited state for example). To describe these systems, it is convenient to introduce the Pauli dimensionless operators proportional to the components of a spin ½:

$$S_i = \frac{\hbar\sigma_i}{2} \quad ; \quad \left[\sigma_i,\sigma_j\right] = 2i\varepsilon_{ijk}\sigma_k$$

which write in a matrix form in the σ_z eigenstates basis::

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad ; \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad ; \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The σ_i's are represented by Hermitian and unitary traceless matrices whose square is the identity operator:

$$\sigma_i^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

One often replaces the spin up and down notation by the binary one (0,1), more usual in quantum information:

$$|0\rangle = |+\rangle_z \quad ; \quad |1\rangle = |-\rangle_z$$

The 3 Pauli operators have the same (+1,-1) spectrum, Their eigenstates are given by::

$$\sigma_z|\pm\rangle_z = \pm|\pm\rangle_z \; or \; \sigma_z|0\rangle = +|0\rangle; \sigma_z|1\rangle = -|1\rangle$$

$$\sigma_x|\pm\rangle_x = \pm\frac{1}{\sqrt{2}}\left(|+\rangle_z \pm |-\rangle_z\right) = \pm\frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$$

$$\sigma_y|\pm\rangle_y = \pm\frac{1}{\sqrt{2}}\left(|+\rangle_z \pm i|-\rangle_z\right) = \pm\frac{1}{\sqrt{2}}\left(|0\rangle \pm i|1\rangle\right)$$

Properties of Pauli operators

Pauli operators are traceless, Hermitian and unitary: $\sigma_i = \sigma_i^\dagger = \sigma_i^{-1} \quad ; \quad \sigma_i^2 = I \qquad T_r\left(\sigma_i\right) = 0$

The product of two Pauli operators is equal to the third one with a +i or –i multiplication: $\sigma_i\sigma_j = i\varepsilon_{ijk}\sigma_k$

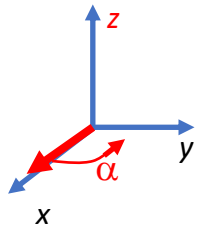
The commutator of two of them is equal to 2i times the third, with the + or – sign depending on the permutation order (commutation rules of angular momentum): $\left[\sigma_i,\sigma_j\right] = 2i\varepsilon_{ijk}\sigma_k$

By linear combination of σ_i and $i\sigma_j$, we get an operator which flips the spin in the k direction: $\sigma_\pm = \frac{1}{2}\left(\sigma_x \pm i\sigma_y\right) \; ; \; \sigma_+ = |+\rangle\langle -| = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \; ; \; \sigma_- = |-\rangle\langle +| = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}$

Any Hermitian operator of trace 1 acting on a spin is a linear combination with real coefficients of the Pauli operators and identity operator (see Lecture 7):

$$O_S = \frac{1}{2}I + \sum_i p_i\sigma_i$$

Spin manipulations: Rotation around Oz



Develop the operator realizing a rotation by angle α around Oz, using the Pauli operator identities:

$$\sigma_i^{2n} = I \; ; \sigma_i^{2n+1} = \sigma_i$$

$$R_z(\alpha) = e^{-i\alpha S_z/\hbar} = e^{-i\alpha\sigma_z/2}$$

$$= 1 - \frac{i\alpha\sigma_z}{2} - \alpha^2 \frac{\sigma_z^2}{2^2 \times 2!} + i\alpha^3 \frac{\sigma_z^3}{2^3 \times 3!} + \ldots.$$

$$= \cos\left(\frac{\alpha}{2}\right) I - i \sin\left(\frac{\alpha}{2}\right) \sigma_z$$

$$= \begin{vmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{vmatrix}$$

Perform this rotation on state $|+\rangle_x$ (spin pointing along Ox)

$$R_z(\alpha)|+\rangle_x = \frac{1}{\sqrt{2}} R_z(\alpha)\left[|+\rangle_z + |-\rangle_z\right]$$

$$= \frac{1}{\sqrt{2}}\left[e^{-i\alpha/2}|+\rangle_z + e^{i\alpha/2}|-\rangle_z\right]$$

Analyse results for 3 values of α:

$$\alpha = \frac{\pi}{2} \; : \; R_z(\frac{\pi}{2})|+\rangle_x = \frac{1}{\sqrt{2}}\left[e^{-i\pi/4}|+\rangle_z + e^{i\pi/4}|-\rangle_z\right]$$

$$= \frac{e^{-i\pi/4}}{\sqrt{2}}\left[|+\rangle_z + i|-\rangle_z\right] = e^{-i\pi/4}|+\rangle_y$$

π/2 rotation around Oz brings spin pointing along Oy

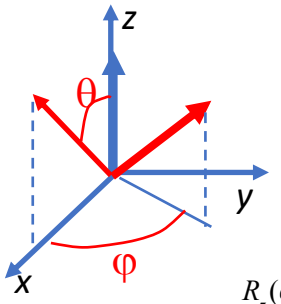
$$\alpha = \pi \; : \; R_z(\pi)|+\rangle_x = \frac{-i}{\sqrt{2}}\left[|+\rangle_z - |-\rangle_z\right] = -i|-\rangle_x$$

π rotation around Oz brings spin pointing along -Ox

$$\alpha = 2\pi \; : \; R_z(2\pi)|+\rangle_x = -|+\rangle_x$$

After 2π rotation, spin comes back to initial state with a sign change (π-phase shift exploited in quantum information ; see later)

Rotate a spin « up » along Oz to get it pointing in direction of polar angles θ,φ



Apply rotation around Oy by angle θ, then rotation by angle φ around Oz. Order of rotations matter (operators do not commute).
The rotations must be applied to the kets from right to left:

$$R_z(\varphi)R_y(\theta) = \left[\cos\left(\frac{\varphi}{2}\right) I - i\sin\left(\frac{\varphi}{2}\right)\sigma_z\right]\left[\cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right)\sigma_y\right]$$

$$= \begin{vmatrix} e^{-i\varphi/2} & \\ & e^{i\varphi/2} \end{vmatrix} \times \begin{vmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{vmatrix} = \begin{vmatrix} e^{-i\varphi/2}\cos\frac{\theta}{2} & -e^{-i\varphi/2}\sin\frac{\theta}{2} \\ e^{i\varphi/2}\sin\frac{\theta}{2} & -e^{i\varphi/2}\cos\frac{\theta}{2} \end{vmatrix}$$

$$R_z(\varphi)R_y(\theta)|+\rangle_z = e^{-i\varphi/2}\cos\frac{\theta}{2}|+\rangle_z + e^{i\varphi/2}\sin\frac{\theta}{2}|-\rangle_z = |+\rangle_{\theta,\varphi}$$

The Bloch sphere representation of a two-level system

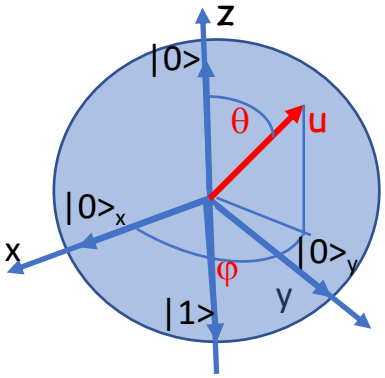
The component along an arbitrary direction u (of polar angles θ,φ) of the pseudo spin is proportional to the linear superposition σ_u of the σ_i matrices:

$$\sigma_u = \cos\theta\sigma_z + \sin\theta\cos\varphi\sigma_x + \sin\theta\sin\varphi\sigma_y = \begin{bmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{bmatrix}$$

The eigenvalues of σ_u are +1 and -1 and the eigenstates noted $|0\rangle_{\theta\varphi}$ and $|1\rangle_{\theta\varphi}$ are linear superpositions of $|0\rangle$ and $|1\rangle$ (see previous page):

$$\sigma_u|0\rangle_{\theta,\varphi} = e^{-i\varphi/2}\cos\frac{\theta}{2}|0\rangle + e^{i\varphi/2}\sin\frac{\theta}{2}|1\rangle$$

$$\sigma_u|1\rangle_{\theta,\varphi} = -e^{-i\varphi/2}\sin\frac{\theta}{2}|0\rangle + e^{i\varphi/2}\cos\frac{\theta}{2}|1\rangle$$



When u is rotated in space, $|0\rangle_{\theta\varphi}$ explores the entire Hilbert space of the pseudo-spin. The tip of this vector belongs to a sphere of radius unity called the Bloch sphere. The Hilbert space is « mirrored » onto this sphere, each point representing a possible superposition of $|0\rangle$ and $|1\rangle$. The north and south poles mirror the $|0\rangle$ and $|1\rangle$ basis states. The eigenstates of σ_x and σ_y are along the equator. Two orthogonal states are at antipodes of the sphere.

Tensor product of Hilbert spaces: quantum description of two spins

The quantum state of a system made of two parts A and B evolves in a global Hilbert space:

$$H_{AB} = H_A \otimes H_B$$

called the tensor product of H_A and H_B. If $|i\rangle_A$ and $|j\rangle_B$ are bases in H_A and H_B, a basis in H_{AB} is given by the tensor products:

$$|i\rangle_A \otimes |j\rangle_B$$

representing the combined system in the state where A is in the state $|i\rangle_A$ and B in $|j\rangle_B$:

Measuring the observable O_A on A without observing B projects the system on the eigenstates of:

$$O_A \otimes I_B$$

tensor product of O_A and the unity operator I_B in H_B. Acting on different systems, the operators $O_A \otimes I_B$ and $O_B \otimes I_A$ commute with each other.

If A and B interact, products of the form.

$$O_A \otimes O_B$$

describe the interaction. Since the tensor product is obviously commutative, we will in the following skip the $\otimes$ symbol and write tensor products as ordinary multiplications.

We can now apply these definitions to an ensemble of two spins noted A and B: Let us define in the tensor product space the 4 state basis:

$$|+\rangle_A|+\rangle_B = |+,+\rangle_{AB} \quad ; \quad |-\rangle_A|-\rangle_B = |-,-\rangle_{AB}$$

$$|+\rangle_A|-\rangle_B = |+,-\rangle_{AB} \quad ; \quad |-\rangle_A|+\rangle_B = |-,+\rangle_{AB}$$

We will now construct another basis in the product Hilbert space which has interesting physical properties.

Additioning two spins ½: the total spin is 1 or 0

By linear combination we construct another orthonormalized basis in the space of the spins

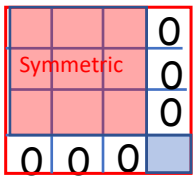
$$|+,+\rangle \quad \frac{1}{\sqrt{2}}\left(|+,-\rangle+|-,+\rangle\right) \quad |-,-\rangle$$

$$\frac{1}{\sqrt{2}}\left(|+,-\rangle-|-,+\rangle\right)$$

The three states of the first line are invariant by permutation of the two spin states (symmetrical states) while the state of the second line changes sign upon permutation (antisymmetrical state).

Consider now the matrix representing the total spin operator $S_A + S_B$ in the above basis. It is obvious that it cannot have matrix elements between states of different permutation symmetry. The action of S_A+S_B on a symmetrical state produces a symmetrical image which is orthogonal to the antisymmetrical state and reciprocally.

Hence, the global spin operator breaks into a 3x3 matrix in the symmetrical subspace and into a 1 diagonal element in the antisymmetrical state:



antisymmetric

Moreover we have obviously:

$$\left(\frac{\sigma_{zA}+\sigma_{zB}}{2}\right)|\pm,\pm\rangle = \pm|\pm,\pm\rangle;$$

$$\left(\frac{\sigma_{zA}+\sigma_{zB}}{2}\right)\left(|+,-\rangle+|-,+\rangle\right)=0$$

Subspace 3x3 of total Spin 1

$$\left(\frac{\sigma_{zA}+\sigma_{zB}}{2}\right)\left(|+,-\rangle-|-,+\rangle\right)=0$$

State of spin 0

Is a quantum state « real »?

In classical physics, a single object can be fabricated with a blueprint by Alice. If she gives it to Bob, he will be able to describe it in details. If he has the required tools, he will also be a able to make an arbitrary number of copies of the object. If Alice distributes copies of the object to independent observers, they will all agree about what they see. All these features are obvious rules of the classical reality.

These rules do not apply in the quantum world. If Alice decides to prepare a quantum state, she will be able to do it with an apparatus designed to measure an observable admitting this state as an eigenstate . For a spin, she will simply measure the relevant σ_u operator and thus project an arbitrary state into any superposition she wishes to obtain:

$$e^{-i\varphi/2}\cos\frac{\theta}{2}|0\rangle + e^{i\varphi/2}\sin\frac{\theta}{2}|1\rangle$$

But if she gives a single particle prepared in this state to Bob, he will not be able to determine the |0> and |1> amplitudes. He may get some information by measuring one pseudo-spin component, but this will erase all further information about the state.

In order to get complete information, Bob must receive a large number of copies of the state. He divides the sample into subsets and measures repeatedly non-commuting observables σ_x,σ_y and σ_z on these subsets, obtaining the mean results yielding θ and φ:

$$\langle\psi|\sigma_z|\psi\rangle = \cos\theta\,;\ \langle\psi|\sigma_x|\psi\rangle = \sin\theta\cos\varphi\,;$$

$$\langle\psi|\sigma_y|\psi\rangle = \sin\theta\sin\varphi$$

This method of state reconstruction is called quantum state tomography:

A quantum state is an "identity" card known only by the « preparer ». To read this card, an observer must have access to a large number of copies and perform a statistical analysis.

"No cloning" theorem

The impossibility to determine the unknown state of a quantum system admits as a corollary the « no-cloning » theorem. Let us demonstrate the impossibility to clone an unknown quantum state by reasoning on spin-like systems.

A cloning machine would perform a unitary operation U transforming the tensor product of an unknown state $|\psi\rangle_A$ and a "neutral" state $|0\rangle_B$ (the "blank sheet" on which A is to be copied) into $|\psi\rangle_A|\psi\rangle_B$. Using the linearity of quantum physics this can be written in two different ways:

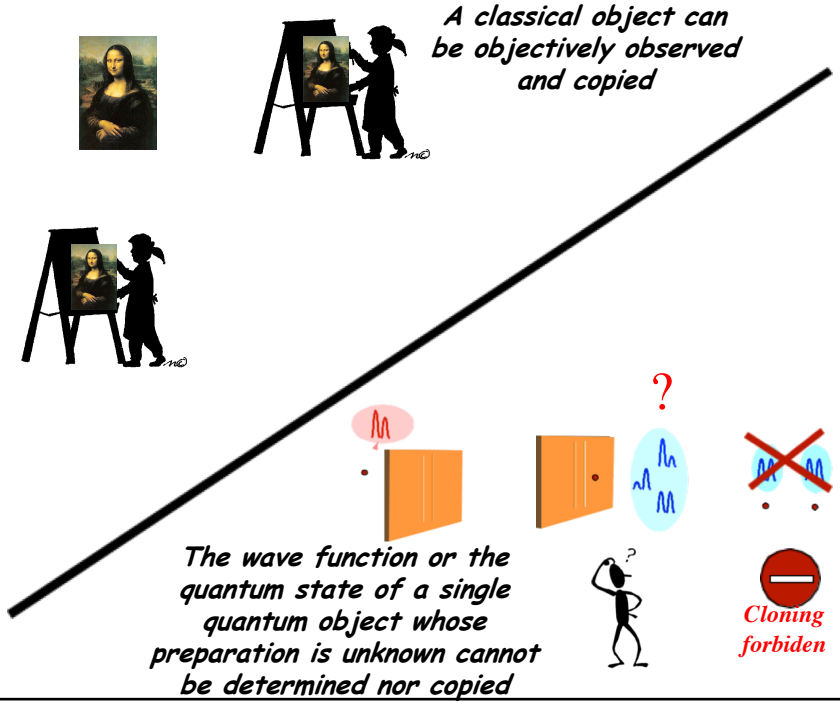
$$U\left(a|0\rangle_A+b|1\rangle_A\right)|0\rangle_B=\left(a|0\rangle_A+b|1\rangle_A\right)\left(a|0\rangle_B+b|1\rangle_B\right)$$

Or:

$$aU|0\rangle_A|0\rangle_B+bU|1\rangle_A|0\rangle_B=a|0\rangle_A|0\rangle_B+b|1\rangle_A|1\rangle_B$$

These two expressions being different (no cross terms in the second one, different amplitudes for $|0\rangle|0\rangle$ and $|1\rangle|1\rangle$) there is an obvious contradiction which shows that quantum cloning is impossible.

If perfect cloning were possible, one could make an arbitrary number of copies of an unknown state and by performing measurements of non-commuting observables on these copies, one could get a complete information on the state, in contradiction with the fundamental property mentionned inthe previous page. Perfect cloning would also make possible superluminal communications (see Lecture 7)



What is the wave function or the quantum state after all?

A mathematical concept useful for calculations which apply to phenomena involving a large number of particles? The wave function makes it possible to compute quantum mechanical averages in systems in which the strange discontinuities predicted by the quantum rules (quantum jumps, state collapse) are not directly observable. Most physicists since the beginning of the 1930's have stopped asking themselves interpretation questions and have used quantum rules in a pragmatic way to understand the properties of light and matter with great success. This is the *« shut up and calculate » era.*

On the other hand, the quantum state was a concept applied in thought experiments to describe hypothetical phenomena at the single particle level, which revealed the strange quantum logic (quantum jump, wave function collapse, entanglement)?. But what reality do these experiments describe? Schrödinger and Einstein were sceptical…

Erwin Schrödinger

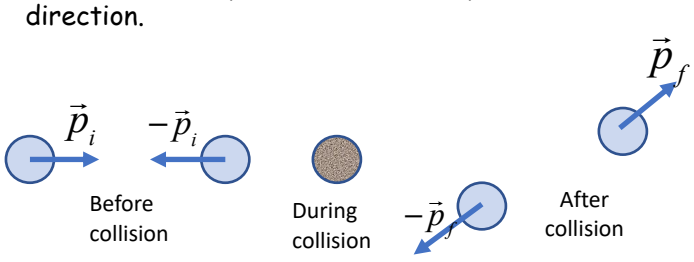
« We never experiment with just one electron or atom or (small) molecule. In thought experiments we sometimes assume we do; this inevitably entails ridiculous consequences » (Schrödinger -1952)

Experiments of the last forty years, notably made possible by lasers, have proven Schrödinger wrong (see next lectures!)

Dealing with identical particles in quantum theory

We have recalled (Lecture 4) how the question of dealing with fundamentally identical particles has been raised early in quantum physics, even before the discovery of the mathematical formalism and the introduction of the concept of wave function and quantum states (Bose-Einstein condensation for Bosons and Pauli exclusion principle for Fermions). In the early days, the question focused on the indistinguishability of identical particles and the consequences it entailed for the calculation of the system's entropy at low temperatures.

As soon as the quantum theory was developed another deep question raised by the treatment of ensemble of identical particles emerged. The superposition principle introduces an ambiguity in the description of the quantum state of an ensemble of particles which needs to be solved for the consistency of the theory. The problem can be recognized for a situation as simple as the collision of two identical particles, for exemple two electrons whose spins are oriented in the same direction.



How to describe initial state?

$|\vec{p}_i,-\vec{p}_i\rangle$ or $|-\vec{p}_i,\vec{p}_i\rangle$?

or $a|\vec{p}_i,-\vec{p}_i\rangle+b|-\vec{p}_i,\vec{p}_i\rangle$

Exchange degeneracy leading to ambiguity in theoretical predictions

Simple example of ambiguity raised by exchange degeneracy and symmetrization postulate

Consider two identical particles evolving in a two dimension Hilbert space, with their pseudo spins oriented in opposite directions, each having a 50% probabilty to point up and down along the quantization direction Oz. We can a priori write the state of the system as:

$$|\psi(\alpha)\rangle = \frac{1}{\sqrt{2}}\left(|+,-\rangle + e^{i\alpha}|-,+\rangle\right)$$

with an arbitrary phase α. This « exchange degeneracy » is not acceptable since it leads to predictions which depend on the choice of α. Suppose for instance that we measure the square of the total spin $(\sigma_1 + \sigma_2)^2$. Its expectation value is:

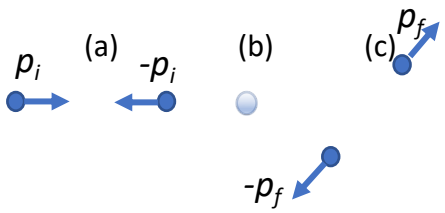
$$\langle\psi(\alpha)|\left(\vec{\sigma}_1 + \vec{\sigma}_2\right)^2|\psi(\alpha)\rangle =$$

$$\langle\psi(\alpha)|\left(\sigma_1^2 + \sigma_2^2 + 2\sigma_{1z}\sigma_{2z} + \sigma_{1+}\sigma_{2-} + \sigma_{1-}\sigma_{2+}\right)|\psi(\alpha)\rangle$$

$$= 4(1+\cos\alpha)$$

Depending on α, we get thus prediction for this expectation value varying from 0 to 8. To avoid this ambiguity, a «symmetrization postulate » is added to the quantum rules: The phase α can only take the values 0 or π. In the first case, the state of the two particles is symmetrical by exchange of the two particles. In the second case, it is antisymmetrical. This rule extends to more than two particles: the state of a system made of identical particles must either remain invariant or change sign by exchange of any couple of particles. In the first case the particles are bosons, in the second case fermions. This postulate leads to physical properties of fermions and bosons confirming those which had been obtained from statistical arguments in the old theory of quanta: the gregarious character of bosons and the exclusion principle for fermions.

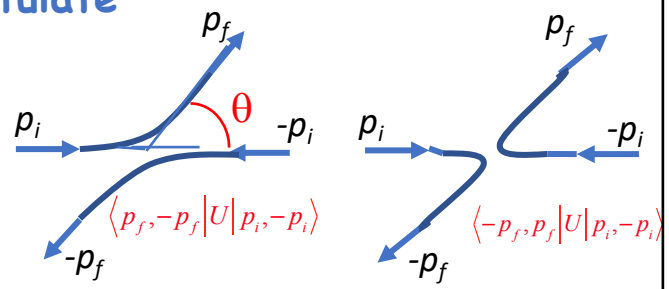
Fermions and Bosons wave function: the symmetrization postulate



Collision of two identical particles in their rest frame: they fly towards each other with opposite velocities (a), their wave packets overlap (b) and they fly away in directions θ and $\pi-\theta$ (c)

$$|\psi\rangle_i = \frac{1}{\sqrt{2}}\left(|p_i,-p_i\rangle + \varepsilon|-p_i,p_i\rangle\right)$$

The system state is either symmetrical by permutation of the two particle (ε=+1, bosons) or antisymmetrical (ε=-1, fermions)



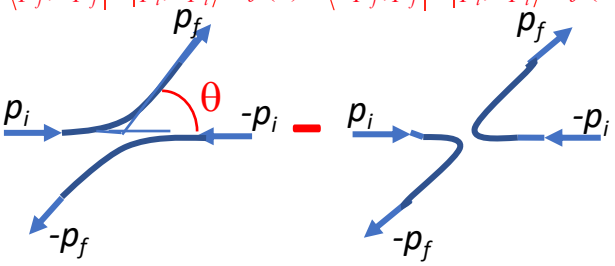
The two interfering amplitudes in the collision between identical particles

The amplitude for the scattering in the directions θ, $\pi-\theta$ is the sum of two interfering terms (U is the unitary evolution operator symmetrical by particle exchange):

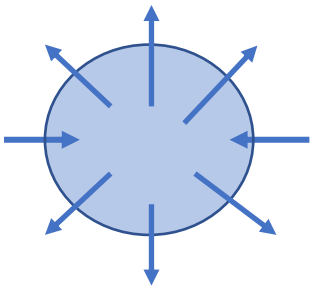
$${}_f\langle\psi|U|\psi\rangle_i = \langle p_f,-p_f|U|p_i,-p_i\rangle + \varepsilon\langle -p_f,p_f|U|p_i,-p_i\rangle$$

Consequences of the antisymmetrization for Fermions

$\langle p_f,-p_f|U|p_i,-p_i\rangle = f(\theta)$ $\langle -p_f,p_f|U|p_i,-p_i\rangle = f(\pi-\theta)$



The two amplitudes interfere with the minus sign: at right angle ($\theta=\pi/2$) the collisional cross section vanishes. This is the case only if there is no way to find out which trajectory was followed by the system. The spins of the fermions should be polarized in the same direction. If they have opposite spins, the two final states could be distinguished and the interference disappears. The fermions then have a finite probability to be scattered at right angle.



For collisions at very low temperatures (de Broglie wavelength larger than the range of the collision potential), the scattering is isotropic, the outgoing particles flying away in a spherically symmetrical s-wave: $f(\theta)$ is then θ independent and the scattering wave vanishes altogether, because the two fermions cannot end-up in the same final state (Pauli exclusion principle). Ultracold polarized fermions fly by each other without interacting. This has important consequences for the manipulation of ultracold Fermi gases.

Antisymmetrization of electron states in atoms: Hund's rule

The antisymmetrization rule for electrons (the global electronic quantum state must change sign for any permutation of couples of electrons) forces them to be in different one-particle states (Pauli exclusion principle). It has also more subtle consequences related to the spin. If two electrons have their spin aligned in same direction, they have an energy smaller than if the electrons have opposite spins (Hund rule). This looks like a magnetic interaction between the electron magnetic moments. It is however an electrostatic interaction orders of magnitude larger.

Consider two electrons occupying orbitals φ and ψ. Their global wave function must be antisymmetrical, which allows two possibilities: either their orbital wave function is antisymmetrical and their spin state is symmetrical or the opposite.

The two electron spin Hilbert space admits a four state basis, three of which (corresponding to a total spin 1) being symmetrical by electron exchange and one (total spin =0) being antisymmetrical:

$$|S\rangle_1 = \frac{1}{\sqrt{2}}\left[|+\rangle_1|-\rangle_2 + |-\rangle_1|+\rangle_2\right] \;;\quad |+\rangle_1|+\rangle_2 \;;\quad |-\rangle_1|-\rangle_2$$

$$|S\rangle_0 = \frac{1}{\sqrt{2}}\left[|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2\right]$$

The state of the two electrons is thus:

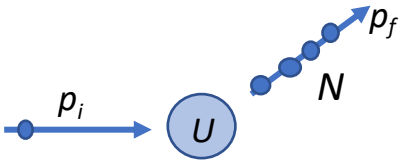
$$|\Psi\rangle_1 = \frac{1}{\sqrt{2}}\left[|\varphi\rangle_1|\psi\rangle_2 - |\psi\rangle_1|\varphi\rangle_2\right]\otimes|S\rangle_1$$

or

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left[|\varphi\rangle_1|\psi\rangle_2 + |\psi\rangle_1|\varphi\rangle_2\right]\otimes|S\rangle_0$$

In the triplet spin 1 states $|\Psi\rangle_1$, the two electrons cannot be at the same point because the orbital wave function vanishes. There is thus less electrostatic repulsion than in the singlet $|\Psi\rangle_0$ state: The system minimizes its energy by aligning its spins. This is the basic explanation for ferromagnetism.

Stimulated scattering of bosons



$$|\psi\rangle_i = \frac{1}{\sqrt{N+1}}\left[|p_i, p_f, p_f \cdots p_f\rangle + |p_f, p_i, p_f \cdots p_f\rangle + \cdots |p_f, p_f, p_f \cdots p_i\rangle\right]$$

$$|\psi\rangle_f = |p_f, p_f, p_f \cdots p_f\rangle$$

$${}_f\langle\psi|U|\psi\rangle_i = \sqrt{N+1}\langle p_f|U|p_i\rangle$$

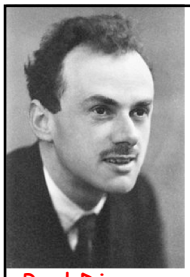
Consider a scattering process in which a boson with momentum p_i impinges on an external potential weak enough so that the probability amplitude for the particle to be scattered spontaneously to a final momentum p_f is small compared to 1 $(U_{fi} \ll 1; U_{ii} \approx 1)$. If N bosons are already present in the final state p_f, the amplitude for the $(N+1)^{th}$ particle to end-up in this state is multiplied by $\sqrt{N+1}$. This bosonic amplification is due to the symmetrization postulate which requires that the N+1 bosonic state be invariant by all particle permutations. It recalls the photon amplification process of stimulated emission discovered by Einstein in 1916. This amplification plays a role in the phenomenon of Bose Einstein condensation: the probability that a particle in the saturated gas phase will fall in the condensed phase is proportional to the number of bosons already in that phase. The more particle are condensed, the faster new particles will join. This is an avalanche process.

Elementary and composite bosons and fermions

The fundamental "indistinguishability" of particles of the same species is an essential feature of quantum physics, which was recongnized even before the maths of quantum physics were discovered. As soon as you associate them with waves, it becomes inconsistent to assign a numerical label to electrons or atoms because it becomes fundamentally impossible to distinguish them by their trajectories. The classical counting is thus a mathematical procedure which must be erased by the symmetrization or antisymmetrization operation. In order to avoid this somewhat cumbersome trick, a formalism of « second quantization » has been developed in which the particles in each quantum state are just represented by their occupation number, which is increased or decreased by creation or annihilation operators. This formalism is particularly useful for photons. The particles appear in general as quantum excitations of their respective fields.

The elementary buiding blocks of matter (electrons, quarks and their « cousins » of the standard model) are fermions. The particles which carry the interactions between them (photons, W and Z carriers of the weak force, gluons and gravitons) are bosons. The fermions have half-integer spins and the bosons integer ones. Besides elementary particles, the bound systems (protons, neutrons made of three quarks, nuclei made of protons and neutrons, atoms made of nuclei and electrons, molecules…) are composite bosons or fermions, depending on the parity of the total number of fundamental fermions which constitute them. This is a consequence of the symmetrization rule. The permutation operator acting on a number of bound fermions will multiply their state by +1 or -1 depending upon the parity of this number. The rules of summation of angular momentum entails that composite fermions have half integer spins and composite bosons integer spins.

According to these rules, protons and neutrons (3 bound quarks) are fermions, the hydrogen atom (1 proton + 1 electron) is a boson as well as the Helium 4 isotope (2 protons, 2 neutrons and 2 electron). Helium 3 is a fermion. We will see that alkali atoms (odd number of electrons) are bosons if they have an odd number of nucleons and fermions if this number is even. In superconducting materials, « Cooper pairs » of electrons behave as bosons.



Paul Dirac
(in 1928)

The theoretical origin of the electron spin and the positron: The Dirac equation

The Schrödinger equation is non-relativistic and thus does not apply to fast moving particles. Marrying the concepts of special relativity with quantum mechanics, Dirac established in 1928 his famous equation, which for a free electron in the absence of electromagnetic fields writes:

$$i\hbar\gamma^{\mu}\partial_{\mu}\psi = mc\psi$$

This equation, which takes a very simple and elegant form, involves three parameters: the Planck constant, the speed of light c and the rest mass m of the electron. The four symbols γ^{μ} (μ=0,1,2,3) are 4x4 matrices (called Dirac matrices) and ∂_{μ} represents the partial derivatives with respect to time (∂_0) and space ($\partial_1,\partial_2,\partial_3$). The summation over μ is implicit. This equation, invariant under Lorentz transformations, describes the electron by a « Dirac spinor » whose four components are position dependent wave functions.

We present here the arguments which have led Dirac to this equation and discuss its physical content: existence of the electron spin and of negative energy states, prediction of the positron and premices of quantum field theories

The requirements of a relativistic quantum mechanical equation for the electron (free particle)

1. The equation must involve space and time in a symmetrical way (same order of derivatives) which is not the case of Schrödinger's equation which is first order in time and second order in spatial derivatives.

2. It should be first order in time derivative (and thus in space derivatives) so that the system's evolution is fully determined by the initial state

3. It should for a free electron satisfy the conservation of the norm of the relativistic energy-momentum quadrivector:

$$E^2 - p^2c^2 - m^2c^4 = 0$$

After identification of E with ihd/dt and p_i (i=x,y,z) with -ihd/dx_i, we must thus get a differential equation linear in time and spatial derivatives whose solution Ψ is an eigenstate with eigenvalue 0 of the above quadratic quantity. The general solution for E writes:

$$E = \pm\sqrt{p^2c^2 + m^2c^4}$$

and we can thus expect that the solution will involve negative energy states whose interpretation will need to be precised.

Dirac matrices and space-time metrics of special relativity

We factorize the quadratic invariant as the product of two linear expressions, with the α_i and β being matrices satisfyng anticommutation relations:

$$E^2 - p^2c^2 - m^2c^4 = \left(E + \vec{\alpha}\vec{p}c + \beta mc^2\right)\left(E - \vec{\alpha}\vec{p}c - \beta mc^2\right)$$

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}I \quad (i,j = x,y,z) \qquad \alpha_i\beta + \beta\alpha_i = 0 \qquad \beta^2 = I$$

The quantum state ψ which satisfies the equation:

$$\left(E - \vec{\alpha}\vec{p}c - \beta mc^2\right)\Psi = 0$$

obviously satisfies also the relativistic invariant second order equation:

$$\left(E^2 - p^2c^2 - m^2c^4\right)\Psi = 0$$

Substituting to E and p the corresponding time and space derivatives, we get the differential equation:

$$\left(i\hbar\frac{\partial}{\partial t} + i\hbar c\vec{\alpha}.\vec{\nabla} - \beta mc^2\right)\Psi = 0$$

We will finally exhibit more clearly the symmetry beween space and time by redefining the Dirac matrices (see next slide).

The Dirac equation in symmetric form

Let us make the notation change:

$$\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}\,;\,\partial_j = \frac{\partial}{\partial x_j}$$

The Dirac equation then writes:

$$\left[i\hbar\left(\partial_0 + \sum_j \alpha_j\partial_j\right) - \beta mc\right]\Psi = 0$$

We then multiply at left by the matrix β and define the new set of matrices γ^μ (μ=0,1,2,3) as:

$$\gamma^0 = \beta\,;\,\gamma^j = \beta\alpha_j$$

which yields Dirac equation in symmetric form (summation on repeated indices):

$$\left[i\hbar\gamma^\mu\partial_\mu - mc\right]\Psi = 0$$

The 4x4 Dirac matrices obey anticommutation relations:

$$\gamma^i\gamma^j + \gamma^j\gamma^i = \beta\alpha_i\beta\alpha_j + \beta\alpha_j\beta\alpha_i$$
$$= -\alpha_i\alpha_j - \alpha_j\alpha_i = -2\delta_{ij}$$
$$\gamma^0\gamma^i + \gamma^i\gamma^0 = \alpha_i + \beta\alpha_i\beta = 0$$
$$\gamma^0\gamma^0 + \gamma^0\gamma^0 = 2I$$

which summarize as:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g_{\mu\nu}I \quad ; \quad g_{\mu\nu} = \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix}$$

We recognize in the anticommutator twice the tensor metrics $g_{\mu\nu}$ of special relativity. Note that the Dirac equation is sometimes simply written in the system unit where c=h=1 as:

$$\left[i\gamma^\mu\partial_\mu - m\right]\Psi = 0$$

Physical content of Dirac's equation (I)

The 4x4 Dirac matrices can be expressed in terms of 2x2 unity and Pauli matrices as:

$$\gamma^0 = \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix}; \gamma^i = \begin{vmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{vmatrix}$$

It is easy to show from the properties of the Pauli operators that this representation of the γ matrices satisfy the required anticommutation relations. The Dirac wave function Ψ is a 4-component « spinor » which we write as:

$$\Psi = \begin{vmatrix} \Psi_1 \\ \Psi_2 \end{vmatrix}$$

where ψ_1 and ψ_2 are two-component functions

We now show that at the non relativistic limit, $|\psi_1|$ is much larger than $|\psi_2|$for positive energy states. It is thus natural to identify ψ_1 with the two component spinor of the Pauli theory described in the first part of this lecture.

With these notations, we easily get:

$$\begin{vmatrix} (E-mc^2)I & c\sigma_i p^i \\ -c\sigma_i p^i & -(E+mc^2)I \end{vmatrix} \begin{vmatrix} \Psi_1 \\ \Psi_2 \end{vmatrix} = 0$$

which is expanded as two coupled equations:

$$\left(E-mc^2\right)\Psi_1 + c\sigma_i p^i \Psi_2 = 0 \; ; -\left(E+mc^2\right)\Psi_2 - c\sigma_i p^i \Psi_1 = 0$$

This allows to express ψ_2 as: $\Psi_2 = -\dfrac{c\sigma_i p^i}{E+mc^2}\Psi_1$

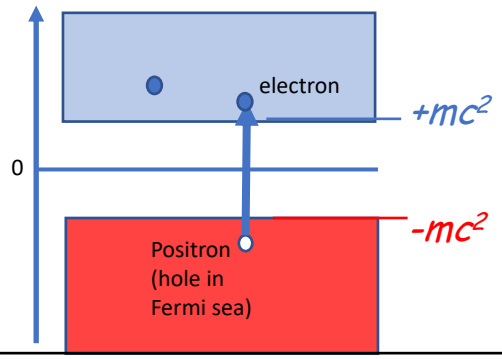
and we find that for positive energy states at the non relativistic limit (E = mc^2+p^2/2m):

$$\frac{|\Psi_2|}{|\Psi_1|} = \frac{pc}{E+mc^2} \sim \frac{mvc}{2mc^2} = \frac{v}{2c} \ll 1$$

Dirac's equation also predicts negative energy states (E= -mc^2-p^2/2m at the non-relativistic limit) whose physical meaning is discussed below. For these states it is easy to show that $|\psi_2/\psi_1| >> 1$.

Physical content of Dirac's equation (II)

From the previous discussion, it is natural to identify ψ_1 as the spinor of Pauli's spin theory. In this sense, Dirac's equation provides the theoretical justification of the Pauli spin model. It remains to understand the properties of the negative energy states which are represented by spinors in which the last two components ψ_2 are dominant. The energy spetrum of an electron is represented by a diagram with a positive energy band above +mc^2 and a negative one below -mc^2, separated by a 2mc^2 forbidden gap:



A positive energy electron cannot spontaneously fall in a negative energy state if all these states are filled (Pauli exclusion principle). Dirac postulates that the vacuum is a Fermi sea full of negative energy electrons. Physical effects should be measured as departure from this vacuum state.

Consider the opposite situation: a photon of energy >2 mc^2 extracts from the Fermi sea a negative energy electron in a kind of photo-electric effect. A hole is left in the Fermi sea and a positive energy electron is created. The hole, a lack of negative energy, is a positive energy excitation of the Fermi sea, a positron. Since a negative elementary charge has disappeared, the positron-hole has a positive charge opposite to that of the electron (and the same mass). Dirac's equation thus predicts that energetic photons can create electron-positron pairs from vacuum. *Note that momentum-energy conservation requires a massive body in vicinity (e.g an atomic nucleus to absorb the recoil).*

The importance of Dirac's equation

Dirac's equation explains the electron spin as an intrinsic relativistic effect. When magnetic field contribution is added to the equation, it predicts that the gyromagnetic ratio of the electron spin (ratio between its magnetic moment and the angular momentum) is twice that of the electron orbital momentum (see next lectures)

Dirac equation has predicted the existence of the positron discovered in 1932 by Anderson.

If one includes the Coulomb potential in the equation, it yields the relativistic spectrum of Hydrogen including the fine structure due to the magnetic interaction between the orbital and spin variables.

It stresses the fact that a relativistic quantum theory cannot be restricted to one-particle. The mass-energy equivalence of relativity combined with the principles allowing for quantum fluctuations makes possible the creation of matter out of vacuum. Particles are quanta of their fields. They can be created or annihilated like photons if enough energy is available (processes essential in high energy physics)

For sake of simplicity, we have restricted ourselves to discuss the Dirac equation of a free electron (no electromagnetic field applied). In the presence of a static Coulomb field or an e-m field described by a quadripotential A_μ, the Dirac equation becomes:

$$\left[\gamma^\mu\left(i\hbar\partial_\mu - qA_\mu\right) - mc\right]\Psi = 0$$

This equation accounts in particular for the magnetic properties of the electron mentionned above. *Note that the vacuum fluctuation of the e-m field are responsible for the "Lamb shift" which adds a supplementary correction to the Hydrogen spectrum as given by Dirac's equation (next lectures)*

Conclusion of Lecture 6

The spin of quantum particles evolving in a two dimension Hilbert space is a simple model for the description of a large variety of quantum systems whose quantum states can be described as superposition of two components (ground and excited state of an atom interacting with light, magnetic moments of nuclei in nuclear magnetic resonance experiments, two state qubits in quantum information science etc…). The Pauli matrices and Bloch sphere representation will be very convenient to describe these situations in future lectures.

We will use the spin formalism in lecture 7 to analyse the concept of quantum entanglement and describe how this concept is related to the notion of complementarity and to the postulates of quantum measurement. We will also describe the principle of operation of qubit quantum gates and show how these gates can be used to generate and analyse entanglement

We will then address the important question of quantum non-locality, the fact that two entangled quantum systems separated by a large distance remain correlated in a way which cannot be explained by classical arguments. We will analyse the famous EPR problem raised in a paper by Einstein in 1935 and show how experiments have been able to demonstrate the existence of these non-classical correlations. We will finally describe the procedure of quantum teleportation which exploits the non-locality of quantum physics to transmit at a distance the quantum state of a qubit.