Sapienza Università di Roma

Admission test – PhD in Physics – XXXV Cycle

Rome, 09/09/2019

Problem Set A

Instructions:
You have four hours to solve 3 (and not more than 3) out of the 6 Exercises below.
Write your solutions only on the sheets provided by the selection committee using a black pen.

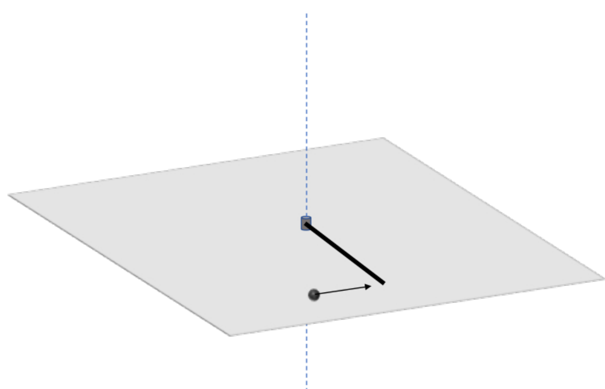
- **Exercise 1: Mechanics – A particle hitting a rod**

  A point mass m is in uniform linear motion, with velocity v_0, on a horizontal plane. The mass hits one end of a rod, of length l and mass M, that is hinged at its other end, and can thus rotate on the same horizontal plane (see drawing below). The collision is completely inelastic.

  a) Determine the angular velocity ω_0 characterizing the system right after the collision.

  b) Assuming there is a constant friction which produces a torque τ in the rotation, determine the number of turns the system performs before stopping.

  [$m = 150$ g ; $v_0 = 20$ m/s ; $l = 28$ cm ; $M = 3.4$ kg ; $\tau = 0.6$ Nm]



- **Exercise 2: Electromagnetism – Charged sphere**

  A sphere of radius R is electrically charged. The charge density ρ is not uniform and depends on the distance from the center r according to the law $\rho = A/r$, where A is a constant.

  a) Evaluate the total charge contained in the sphere.

  b) Determine the dependence on r of the electric field both inside and outside the sphere.

  c) Evaluate the total electrostatic energy of the system.

  [$R = 2.5$ cm ; $A = 0.0027$ C/m^2; $\epsilon_0 = 8.8 \times 10^{-12}$ C^2/Nm^2]

- **Exercise 3: Relativity – Supernova and neutrinos**

  During the famous supernova SN1987A (approximate distance from Earth: $d = 1.7 \times 10^5$ light-years), some neutrinos with energies in the range $20 - 30\,\text{MeV}$ had been detected within a time interval of about $\Delta t = 10\,\text{s}$.

  a) Assuming for simplicity that all neutrinos had been produced identical at the same time during the supernova and neglectig neutrino oscillations, compute the maximum value of the neutrino mass compatible with the data.
  Hint: the computation could be simplified by making some approximation on the neutrino mass.

  b) One of the possible channels of neutrino detection involves an elastic scattering with electrons. Compute the center of mass energy for the scattering of the most energetic neutrinos (energy $E_\nu \sim 30\,\text{MeV}$) with an electron at rest [electron mass $m_e \approx 0.5\,\text{MeV}$].

- **Exercise 4: Analytical Mechanics – Lagrangian and Hamiltonian**

  A system comprising of two degrees of freedom ($x = x(t)$ and $y = y(t)$, respectively) satisfies the following system of equations

$$\ddot{x} + \omega_x^2 x = \gamma y^2 x \tag{1}$$
$$\ddot{y} + \omega_y^2 y = \gamma x^2 y \tag{2}$$

  where γ, ω_x, and ω_y are real constants.

  a) Find the Lagrangian of the system.

  b) Find the corresponding Hamiltonian.

  c) Prove that the Hamiltonian is a conserved quantity and find a canonical transformation.

- **Exercise 5: Atomic physics – Chlorine atom**

  Chlorine (Cl) has the following ground-state electronic configuration $(1s)^2(2s)^2(2p)^6(3s)^2(3p)^5$. The energy levels of the ground state and of the $(1s)^2(2s)^2(2p)^6(3s)^2(3p)^4(4s)$ excited state, measured starting from the ground state, are reported in the following table:

Electronic configuration	Term	J	Energy (cm^{-1})
[Ne]$3s^23p^5$			0
			881
[Ne]$3s^23p^44s$			71954
			72484
			72823
	^2D	5/2	74221
		3/2	74861
	^2P	3/2	84116
		1/2	84127
	^2S	1/2	88351

  a) Complete the table indicating the spectroscopic terms, spin degeneracy, and J, placing them in the correct order.

  b) Suppose to excite the Cl atoms with electrons of energy between $74000\,\text{cm}^{-1}$ and $75000\,\text{cm}^{-1}$. Which emission lines will be measured in the dipole approximation?

- **Exercise 6: Statistics – Cosmic ray flux**

  The average cosmic ray flux at the sea-level for a horizontal detector is roughly equal to ϕ_0 =1 cm^{-2} min^{-1}. We assume this number is known without any uncertainty.

  a) Evaluate the minimum side of a horizontal square detector such that the probability to count one or more than one event per second is larger than 90%.

  b) Evaluate how much time a detector of such dimensions needs to run to reach a statistical uncertainty smaller than 1%.

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Problem Set B

Instructions:
You have four hours to solve 3 (and not more than 3) out of the 6 Exercises below.
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- **Exercise 1: Thermodynamics – Gas in a a piston-cylinder assembly**

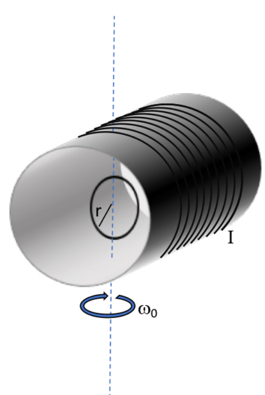
  One mole of an ideal monoatomic gas is held in a piston-cylinder assembly, which experiences a constant ambient external pressure P_0. Consider the piston to be massless and to move without friction. The piston-cylinder assembly is thermally insulated. At the beginning, a mass M is placed on the piston. The gas is in equilibrium, has a volume V_A, and is at temperature $T_A = 300$ K. As the mass M is removed, the gas expands instantaneously (e.g. the piston rises) and, after some time, reaches a new equilibrium position B, with $V_B = 4V_A$.

  a) Calculate the value of the temperature T_B.
  b) Once at T_B, the gas is used as thermal reservoir to a composite reversible Carnot engine operating between three thermal reservoirs: for each cycle, the engine absorbs 10 J from a reservoir at $T_1 = 400$ K and 10 J of heat is transferred to the T_B reservoir. Calculate the amount of heat exchanged with the third thermal reservoir at $T_2 = 300$ K.

- **Exercise 2: Electromagnetism – Loop inside a solenoid**

  A conductive circular loop of radius r and total resistance R is fully contained in a solenoid of length l and radius $\ll l$ made of N loops. The current in the solenoid is I. The circular loop is initially kept perpendicular to the axis of the solenoid as shown in the drawing below and can be put in rotation around an axis perpendicular to that of the solenoid. Starting from $t = 0$, the loop is kept in rotation with constant angular velocity ω.

  a) Evaluate the flux of the magnetic field through the loop at $t = 0$.
  b) Determine the expression of the current in the loop as a function of time.
  c) Evaluate the total charge flown across any section of the loop wire once the first half-turn of the loop is completed.



[$r = 15\,\mathrm{cm}$; $l = 2\,\mathrm{m}$; $N = 2000$; $R = 50\,\Omega$; $I = 1.2\,\mathrm{mA}$; $\omega = 30\,\mathrm{rad/s}$; $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{Tm/A}$]

- **Exercise 3: Relativity – Reference frames and scattering**

  A reference frame O' is moving with constant speed $v = 0.999\,c$ along the x direction relative to a reference frame O. In O a photon, with frequency $\nu = 5 \times 10^{19}$ Hz, moves along a direction identified by an angle θ relative to the x axis.

  a) Find the angle α for which the frequency of the photon is the same in both frames.

  b) The photon is scattered by an electron at rest. In the scattering the photon loses one third of its initial energy. Compute the longitudinal and transversal components of the velocity vector of the scattered electron.
  Hint: Compton formula: $\lambda_f = \lambda + \lambda_c(1 + \cos\theta)$, *where* $\lambda_c = h/(m_e c)$.
  [$m_e = 9.1 \times 10^{-31}$ kg ; $h = 6.62 \times 10^{-34}\text{m}^2$ kg s^{-1}]

- **Exercise 4: Quantum mechanics – Perturbed harmonic oscillator**

  Consider a quantum harmonic oscillator in one spatial dimension x, described by the following Hamiltonian

  $$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \eta x^4\,, \tag{3}$$

  where $m, \eta > 0$. The term proportional to η is a small perturbation.

  a) Consider first the unperturbed oscillator by setting $\eta = 0$. The fundamental eigenfunction is $\psi_0(x) \propto e^{-\alpha x^2/2}$ with $\alpha = m\omega/\hbar$. Compute the corresponding eigenvalue E_0 and the normalization of the eigenfunction.
  Hint: $\int_{-\infty}^{+\infty} dy e^{-y^2} y^{2n} = \frac{(2n)!}{4^n n!}\sqrt{\pi}$ *for any non-negative integer* n.

  b) Using time-independent perturbation theory, compute the correction ΔE to the unperturbed ground-state energy E_0 to first order in the perturbation η.

  c) Write the Lagrangian L associated to the above Hamiltonian.

- **Exercise 5: Atomic physics – Spectroscopic series**

  A spectroscopic series is measured with a poor spectral resolution on Hydrogen atoms and the following lines of the emission spectra (in cm^{-1}) are found: 15230, 20565, 23030, 24370, 25515.

  a) Identify the final energy level (the lower one, to which the electron decays) and thus which series was measured.

  b) Calculate how the transition energies would change if, instead of Hydrogen, one measures the same series for Deuterium. Which would be the minimum spectral resolution needed to distinguish among H and D?

  [$R_H = 109677\,\text{cm}^{-1}$; the mass ratio between a proton and an electron is $m_p/m_e \approx 1836$]

- **Exercise 6: Statistics – Photo-detectors and failure probability**

  W need to install 10 big photo-detectors for an experiment. A failure probability p_f =0.8% for each photo-detector is estimated by the company producing them.

  a) How much is the probability that all photo-detectors are working if we assume the company's estimate to be correct?

  b) The company claims that, given the established conditions, the probability that at least 9 out of 10 photo-detectors are working exceeds 99%. Is the company right? Motivate the answer.

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Problem Set C

Instructions:
You have four hours to solve 3 (and not more than 3) out of the 6 Exercises below.
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- **Exercise 1: Mechanics – Wire on a rotating cylinder**

  A massless un-stretchable wire is wound around a cylinder with base radius r and negligible mass. The cylinder can rotate freely about its horizontal axis a that is kept fixed (see left drawing below). One end of the wire is fixed to the cylinder, the other one to a point mass M. A rod with length l and negligible mass is fixed orthogonal to the cylinder's axis and a point mass $m < M$ is attached to its loose end.

  a) Identify the angle that the rod forms with the vertical axis at equilibrium and the minimum value of l that allows the system to be in equilibrium.

  b) Determine how the equilibrium position is modified if the mass m is instead distributed uniformly along the rod of length l.

  [*Hint: the moment of inertia of a uniform rod with mass m and length l relative to its center of mass is* $I = \frac{1}{12}ml^2$.]

  c) In the last case, assuming to perturb the system from its stable equilibrium position, evaluate the period of the motion in the approximation of small oscillations.

  [$M = 1.3$ kg ; $m = 450$ g ; $r = 16$ cm ; $l = 65$ cm]

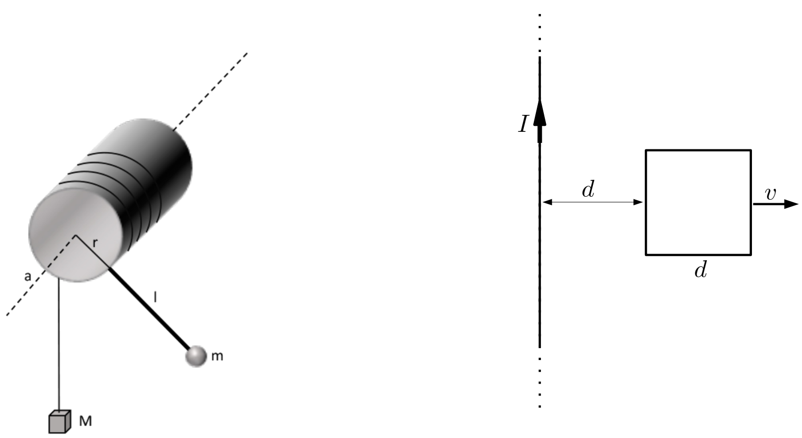

Figure 1: Drawing for Exercise 1 (left) and for Exercise 2 (right).

- **Exercise 2: Electromagnetism – Moving loop and wire**

  A constant current I flows on a metallic wire of infinite length. A metallic square loop of side d and overall electrical resistance R is located close to the wire with two sides parallel to it (as shown in the right drawing above). Initially the loop side closest to the wire is kept at a distance d from the wire. At a given time, t=0, the loop starts to move at constant speed v.

  a) Evaluate the flux of the magnetic field through the loop in the initial conditions.

  b) Determine the expression of the induced electromotive force as a function of time.

  c) Evaluate the current circulating in the loop when the side closest to the wire is at a distance equal to $2d$ from the wire.

  [$I = 8$ mA ; $d = 16$ cm ; $R = 50\,\Omega$; $v = 10\,$m/s; $\mu_0 = 4\pi \times 10^{-7}$ Tm/A]

- **Exercise 3. Relativity – Pion decays**

  A charged pion beam is stopped on a target where the pions decay. We consider only two possible pio-decay modes, namely $\pi \to \mu\nu_\mu$ and $\pi \to e\nu_e$.

  a) Evaluate the kinetic energy of the electron, of the muon, and of the neutrinos produced in the two decays.
  b) Assuming a spherical detector of radius R, evaluate the time needed for the muon and for the neutrino to reach the detector.
  c) A very rare decay is $\pi^- \to \pi^0 + e^- + \bar{\nu}_e$. In this case determine the maximum energy of the neutrino.

  [$m_\pi = 139.6$ MeV ; $m_\mu = 105.7$ MeV ; $m_e = 0.511$ MeV ; $m_\nu = 0$; $R = 3$m]

- **Exercise 4: Quantum mechanics – Scattering off a potential**

  Consider the one-dimensional, non-relativistic, quantum mechanical scattering of a particle (with mass m and energy E) off a potential $V(x)$ such that

  $$V(x) = \begin{cases} +\infty & x < 0 \\ 0 & 0 < x < L \\ V_1 & L < x < 2L \\ 0 & x > 2L \end{cases} \tag{4}$$

  where V_1 and L are positive constants. The waveform is normalized such that the amplitude of the incident wave coming from the right is unity.

  a) Compute the amplitude of the reflected wave.
  b) In the above case, what is the fraction of reflected flux relative to the incident one?
  c) The infinite potential barrier is removed. In such case, compute the probability to find the particle at $x < 0$.

  Hint: the computation of points a) and c) above can be simplified by assuming $V_1 \gg 2mE/\hbar^2$.

- **Exercise 5: Atomic physics – Carbon atom**

  Carbon presents the following ground-state electronic configuration [He]$(2s)^2(2p)^2$.

  a) Write the possible energy levels arising from this configuration and identify the ground state.
  b) Suppose to excite one electron in the [He]$(2s)^2(2p)(3s)$ configuration. Write the spectroscopic terms associated to this electronic configuration, knowing that one corresponds to a binding energy of $-3.9\,$eV and one to $-3.7\,$eV. Which emission lines will be measured in a decay to the ground state ($-11.3\,$eV) if all the states are full?
  c) Which would be the lower energy level arising instead from the [He]$(2s)(2p)^3$ configuration?

  Hint: consider degenerate levels with different J, and assume Hund's rule valid also for excited states.

- **Exercise 6: Statistics – Gaussian distribution**

  Consider two uncorrelated random variables x and y each distributed according to a Gaussian distribution. They have the same mean value $\mu_x = \mu_y = 1.3$ but different variances $\sigma_x = 0.3$ and $\sigma_y = 0.5$, respectively.

  a) Calculate: the probability of: having $x > 2$; the probability of having $y > 2$
  b) Calculate the probability of having $x > 2$ and $y > 2$ simultaneously
  c) How can a χ^2 variable be built starting from these two variables?

  Gaussian tables are provided in attachment.