Written Test A

[Solve three out of the following five problems.]

Problem 1. A spinless particle is described by the wave function

$$\psi = B(x + y + 2z)e^{-\sqrt{x^2+y^2+z^2}}$$

where B is a constant.

1. Determine the total angular momentum J of the particle.
2. Determine the probabilities of finding it with all possible J_z values. The spherical harmonics listed may be of use.

$$\left[\quad Y_0^0 = \frac{1}{\sqrt{4\pi}}, \quad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_1^1 = -\sqrt{\frac{3}{8\pi}}\,e^{i\phi}\sin\theta, \quad Y_1^{-1} = \sqrt{\frac{3}{8\pi}}\,e^{-i\phi}\sin\theta, \right.$$
$$Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1), \quad Y_2^{-1} = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{-i\phi}, \quad Y_2^{-2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{-2i\phi},$$
$$\left. Y_2^2 = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{2i\phi}, \quad Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi} \quad \right]$$

Problem 2. A mass m slides on a horizontal frictionless track. It is connected to a spring fastened to a wall. Initially the amplitude of the oscillation is A_1 and the elastic constant of the spring is k_1.

1. Find the phase trajectory in the (q, p) plane where q is the position and p is the momentum.
2. The elastic constant slowly (with respect to the period of oscillation) decreases to reach a second value $k_2 < k_1$. What is the new amplitude of oscillation A_2 given that the area enclosed by the phase trajectory is an invariant of motion, in the 'adiabatic' conditions described?

Problem 3. In a asymmetric collider, 9 GeV electrons make head on collisions on 3 GeV positrons. The interaction produces a $B\bar{B}$ meson pair.

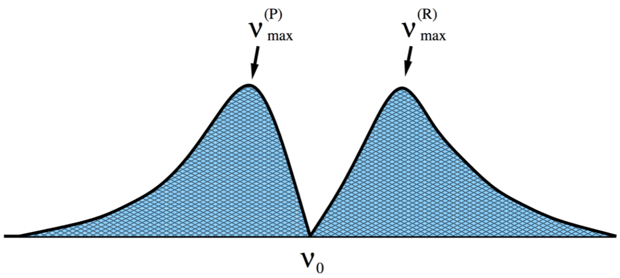
1. What are the total energy of the system in center of mass (CM) frame and the CM velocity in the laboratory (LAB) frame?
2. What are the longitudinal and transverse momenta of the B mesons in the LAB frame assuming they recoil in a direction perpendicular to the beam line in the center of mass frame?
3. What is the probability of a B meson to decay within 300 μm in the LAB frame?

$$\left[\quad m_e = 511\ \text{keV/c}^2, \quad m_B = 5\ \text{GeV/c}^2, \quad \tau_B = 1.5\ \text{ps} \quad \right]$$

Problem 4. A cylinder with adiabatic walls is closed at the top by an adiabatic mobile piston. The cylinder contains a small piece of material with thermal capacity C and n moles of a monoatomic perfect gas. The whole system is initially at equilibrium at temperature T_0 with P_0 the initial pressure of the gas. Assume that the external atmospheric pressure is negligible.

1. Calculate the relationship between P and T for a transformation in which the pressure is increased reversibly acting on the piston. Calculate the final temperature if the final pressure is $2P_0$.
2. If instead the force acting on the piston is doubled *instantaneously*, calculate the temperature of the system at thermal equilibrium in the final state.
3. Calculate the entropy variation for the system and the universe in the two transformations described above.

Problem 5. The roto-vibrational absorption spectrum of a diatomic molecule at $T = 135$ K is shown in figure. This spectrum was collected with a low-resolving-power spectrophotometer such that



the individual rotational lines are not resolved. The two frequencies shown in figure, for which the absorption intensity is maximum, correspond to $h\nu_{\max}^{(P)} = 1376$ cm^{-1} and $h\nu_{\max}^{(R)} = 1451$ cm^{-1}.

1. Write the energy of the absorption lines for the P- and R-branches as a function of the rotational quantum number J of the starting level of the transition.
2. Calculate the approximate value of the rotational constant B in cm^{-1} and the value $J_{\max}$ corresponding to the level with the highest occupation number at the given temperature.
3. Calculate the approximate value of the vibrational energy $h\nu_0$ in cm^{-1}.

Written Test B

[Solve three out of the following five problems.]

Problem 1. Consider the following quantum Hamiltonian

$$H = H_0 + H_1 = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 + H_1$$

where

$$H_1 = \lambda \left(-i\frac{d}{dx}\right)$$

and λ is a real parameter.

1. H_0 is the Hamiltonian of a linear oscillator. Write H_1 in terms of $\hbar$, the mass of the oscillator m and its characteristic frequency ω (which are set to 1 in formulae given above).
2. Assuming H_1 to be a small perturbation, compute the first order correction to the unperturbed ground state energy.
3. Determine the ground state wave function of H and its energy levels. It may be convenient to use the following annihilation/creation operators.

Problem 2. A partially filled glass of water rotates around its symmetry axis with an angular frequency $\omega = 2$ rps (rounds per second).

1. Determine the shape of the water surface in terms of a function $f(r)$ of the distance r of each point on the surface from the rotation axis (water is assumed not to overflow out of the glass).
2. The point on the water surface through the symmetry axis drops down by a distance h upon rotation. Determine the value of h in the case of a cylindrical glass of radius $a = 5$ cm.
3. An observer looking a *still* glass of water from above will have the impression that its depth is less, more or equal than the true depth? In the first two cases, which factor does the depth appear to be decreased/increased by?
   [The refractive index of water is $n \simeq 4/3$.]

Problem 3. A 3 GeV kaon K^+ beam is sent to a decay pipe 200 m long. The K^+ decays into $K^+ \to \mu^+\nu$.

1. Determine the fraction of K^+ decaying within the tunnel.

2. Determine the maximum neutrino momentum in the laboratory frame.

3. The neutrinos collide on a fixed target rich in neutrons. Could they produce the reaction $\nu + n \to \mu^- + \Lambda_c^+$?

$$\left[\begin{array}{l} m_\mu = 105.7\ \mathrm{MeV/c^2},\ m_{K+} = 493.7\ \mathrm{MeV/c^2},\ \tau_{K+} = 12.4\ \mathrm{ns}, \\ m_n = 939.5\ \mathrm{MeV/c^2},\ m_{\Lambda_c^+} = 2286.3\ \mathrm{MeV/c^2} \end{array}\right]$$

Problem 4. A thermally insulated system consists of two parts, A and B, of an ideal gas separated by a thermally conducting and movable (frictionless) partition. Initially the partition is clamped and the thermodynamic variables in the two sectors are: $V_A = 2V_0$; $P_A = 3P_0$; $T_A = T_0$ and $V_B = V_0$; $P_B = P_0$; $T_B = T_0$. The partition is then allowed to move without gas mixing.

1. Calculate the variation of the internal energy and P_f, T_f of the final equilibrium state of the system.

2. Calculate the change of entropy of the system and of the universe.

Problem 5. A beam of electromagnetic radiation with wave numbers ranging from 2470 cm^{-1} to 2580 cm^{-1} is directed towards a transparent vessel containing gaseous HBr at $T = 450$ K. The following absorption lines are observed: 2478 cm^{-1}, 2494 cm^{-1}, 2510 cm^{-1}, 2526 cm^{-1}, 2542 cm^{-1}, 2574 cm^{-1}.

1. Which kind of absorption spectrum is observed? What kind of spectroscopic information can be obtained?

2. What is the most intense line observed?

3. What spectrum can be observed using the same electromagnetic radiation but substituting H with D? (Note: the atomic mass of Br is 80 a.m.u.)

Written Test C

[Solve three out of the following five problems.]

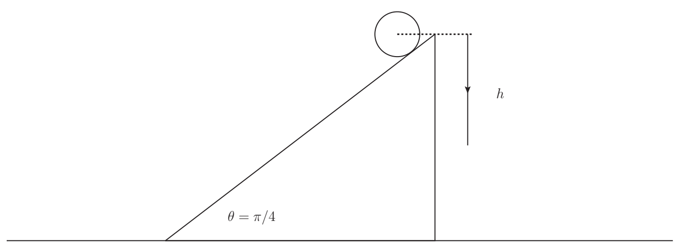
Problem 1. Consider the Hamiltonian of the linear oscillator

$$H = a^\dagger a + \frac{1}{2} \quad \text{where} \quad a^\dagger = \frac{x - ip}{\sqrt{2}}, \qquad a = \frac{x + ip}{\sqrt{2}}$$

1. Write the annihilation and creation operators, and H itself, in terms of $\hbar$, the mass of the oscillator m and its characteristic frequency ω (which are set to 1 in formulae given above).
2. Find the *normalized* eigenfunctions ψ_0 and ψ_1 corresponding to the ground state and to the first excited state of H.
3. The equation $a^\dagger \psi = 0$ has a solution, find it. Can ψ be considered as the state of maximum possible energy of H?

Problem 2. Consider a solid cylinder of mass m and radius r on the inclined face of a wedge of mass M. The wedge can slide on a frictionless horizontal surface. Initially the whole system is at rest. The cylinder descends by a distance h.

1. How far has the wedge moved if the cylinder rolls without slipping and if it slips without friction?
2. In which of the two cases does the cylinder reach the bottom earlier?
3. What is the work done on the wedge when the cylinder slips without friction?



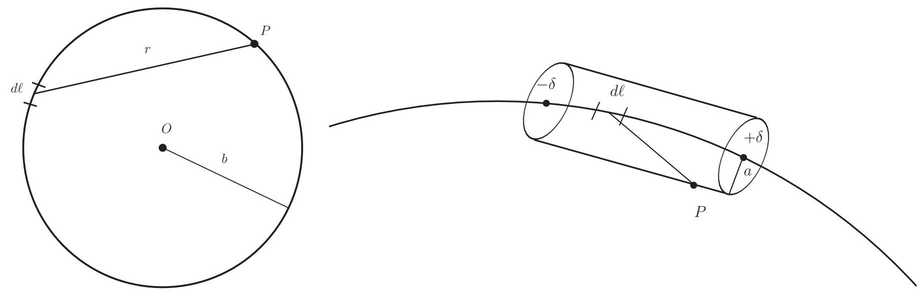
Problem 3. In the annihilation of anti-protons $\bar{p}$ with momentum $p = 10$ GeV/c on protons at rest, $\Lambda\bar{\Lambda}$ pairs are produced.

1. What is the Λ momentum in the laboratory (LAB) frame in the case in which the $\Lambda\bar{\Lambda}$ pair recoils back-to-back in the center of mass frame in a direction perpendicular to the beam line?
2. What is the Λ flight angle in the LAB frame with respect to the beam line?

3. What is the decay fraction within 80 cm?

$$\left[\quad m_p = 0.938 \text{ GeV/c}^2, \quad m_\Lambda = 1.116 \text{ GeV/c}^2, \quad \tau_\Lambda = 260 \text{ ps} \quad \right]$$

Problem 4. Consider a ring of radius b made of a thin conducting wire, with a section of radius $a \ll b$ (see figure). We want to compute the capacitance of the ring.



Since the wire is thin, the field at the surface of the ring, for large b, is almost equal to the field due to charges distributed along the axis of the wire, with a linear density $\lambda = Q/2\pi b$.

1. Compute the potential at point P on the surface, due to far away segments of charge $\lambda\, d\ell$ — consider only distances $r > 2\delta$ (where δ is some value $\delta \gg a$). Neglect the section of the ring. See figure on the left.

2. Compute the potential in P due to closeby segments of charge $\lambda\, d\ell$ within the interval $-\delta < \ell < \delta$ ($\delta \gg a$ and $\delta \ll b$). In this case a cannot be neglected. See figure on the right. Notice that for $r < 2\delta$ the segment of the ring may be considered as straight.

3. Sum the two contributions and determine the capacitance of the ring. The following integrals may be of use.

$$\left[\quad \int \frac{dx}{\sin(x/2)} = 2 \ln \tan(x/4) + C, \qquad \int \frac{dx}{\sqrt{x^2+1}} = \ln(x + \sqrt{x^2+1}) + C \quad \right]$$

Problem 5. The roto-vibrational spectrum of a diatomic molecule having a vibrational frequency $h\nu_0 =$ 2143 cm^{-1} is collected over the 2140 cm^{-1} e 2160 cm^{-1} frequency range. Assume a rotational constant $B = 1.9$ cm^{-1}.

1. How many lines are observed? If the equilibrium distance is $R_0 = 1.13$ Å, calculate the reduced mass of the molecule.

2. If the measurement is done at $T = 265$ K, at which frequency corresponds to the most intense line observed? Draw a qualitative scheme of the portion of the measured roto-vibrational spectrum.