Quantum particles in quantum spacetime

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Dedicated to
Arianna
Abstract

Non-commutative spacetime theories are an effective-theory approach to the description of quantum gravity effects on particle propagation. Some basic approximations of classical (pre-quantum gravity) theories are maintained, like for example the presence of a flat background spacetime and the smallness of gravitational attraction between microscopic bodies.

Here I aim to study the interplay between spacetime (gravitational) dynamics and quantum effects, in an attempt to picture modifications on particles localization which we expect as common in any quantum gravity candidate theory. In particular we expect spacetime to develop a degree of fuzziness for scales approaching the Planck length - or equivalently for energies approaching the Planck mass.

In this thesis I will study the problem of representing non-commutative spacetime theories on a satisfactory Hilbert space of states, and to do so I will exploit recent progress in a covariant formulation of quantum mechanics. This reformulation of quantum mechanics in a more symmetric fashion is the perfect environment to describe the relativistic dynamics of quantum, free particles, and will be the ideal setting for us to introduce non-trivial commutation relations between spacetime coordinate operators.

The covariant quantum mechanics formalism is applied to two of the most studied examples of non-commutative spacetime: $\kappa$-Minkowski and Snyder spacetime. In both cases I will find results which are more satisfactory than the ones present in literature before, both from a phenomenological point of view - in $\kappa$-Minkowski I will predict phenomena impossible to describe in other formalism - and in terms of mathematical rigour. As a consequence of the work described in this thesis a number of heuristic hypothesis about quantum spacetime are revealed to be not well-founded, and it is shown how they will need a critical revision.
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Introduction

One of the most significant challenges for modern theoretical physics is a satisfactory description of the regime of physics where both gravitational and quantum phenomena have to be taken into account. This is often referred to as "Quantum Gravity" problem, and this thesis work will focus on some particular aspects of this problem. Quantum Mechanics and General Relativity are very successful by themselves, although not completely free of conceptual problems. They are at the present time the best tools we have in understanding both the dynamics of microscopic particles and the most vast and far away components of our universe.

The modern incarnation of quantum theory, the Standard Model of particle physics, has both revolutionized our way of understanding the microscopic world and given us the tools to make spectacular predictions about particles dynamics. Our description of quantum effects has been tested down to lengths of order $10^{-20}$ m up to the micrometer scale [1], and some of its predictions have been tested with a precision of 1 part in a billion. General relativity on the other hand revolutionized our ideas of space and time, and gave us a whole new method of looking at physical theories. On the experimental side it has been confirmed on distances going from $10^{-6}$m to astrophysical scales [2].

Even though these two theories are extremely successful on their own, it should be noted that they are applied in describing two different regimes of our world. Quantum mechanics assumes the existence of a background, absolute spacetime, on which fields are defined and where their evolution takes place without ever having any consequences on the spacetime itself. General relativity makes the opposite of this approximation one of its basic assumptions. It is the theory of spacetime dynamics, and one of its main beauties is of being free of an absolute background on which to define dynamical objects. On the other hand General relativity is based on a classical description of the world, in which every physical observable has a well defined value and a deterministic evolution.

We expect that both of them are just approximations, since on the one hand spacetime truly is dynamical and on the other hand our physical observables must have quantum properties. So both these theories (perfectly working, until now) are long-distance and low-energy approximations of a more fundamental theory embracing (or reproducing in the right limit) properties from both the quantum and the general relativistic regime. The problem of finding this more fundamental theory is made both more difficult and "less pressing" for mainstream physics because of the lack of an experimentally reachable regime in which both theories could be tested at the same time. In such a situation neither quantum mechanics nor general relativity give reliable prediction, so what we could see in a similar observation would both
surprise us and guide us toward the finding of a more complete theory. This lack of "experimental problems" does not really reduce the significance of the quantum gravity problem, since it is troubling that there are conceivable (though, so far, practically undoable) experiments for which we are not able to predict the outcome, at the best of our current knowledge. The energy before which we expect our ability to make predictions not to be valid anymore is obtained combining the fundamental constants of both general relativity and quantum mechanics:

\[ E_P = \sqrt{\frac{c^5 \hbar}{G}} \approx 1,221 \times 10^{28} \text{eV} \]

Thus, for example, we will have no reliable description of a particle collision in which the involved energies are comparable with that value, until we find a more general theoretical setting than quantum mechanics and general relativity. The reason is that at those energies the gravitational recoil of particles cannot be neglected, so dynamics of spacetime becomes important, general relativity is necessary in the description of the process, and the notion of absolute spacetime loses meaning. But without such a notion we are not able to define a sensible quantum field theory, so we cannot say what happens to our particles.

In this thesis I don’t try to find a new, fundamental theory of quantum gravity which would merge quantum mechanics and general relativity in a single setting. An example of such a theory is Loop Quantum Gravity [3], which is a quantization of the gravitational field, expressed in the suitable variables. Such 'fundamental' theories are however more challenging from a phenomenological and calculational point of view, although some recent progress has been made on this side [4]. Here I take an effective approach, which introduces new structures of the type quantum-gravity might plausibly require in a setting that preserves much of the simplicity of our current theories. The new structures I am most interested in exploring are those connected with the possibility that combining quantum mechanics and general relativity the geometry of spacetime may require a new kind of description.

Both quantum mechanics and general relativity are based on the concept of classical spacetime, i.e. they admit the abstraction of absolute localization of spacetime points. This sharp localization is however attained in very different - and in fact conflicting - ways. In quantum field theory the ideal procedure to define spacetime points involves particles of infinite mass (or energy); quantum fluctuations of particles position decrease with increasing particles masses, so it is possible to consider a limiting procedure in which one considers particles of arbitrarily high mass, and uses them to build a reference frame which identifies spacetime points. It is crucial, in this limit, the existence of a spacetime background which does not get deformed by the presence of very massive particles. In general relativity the situation is exactly the opposite: since spacetime is classical a typical collision event will identify sharply a spacetime point, however in the case of very massive particles one stumbles upon the issues connected with the singularities of general relativity. A collision among particles of very high mass may not result in an operative localization procedure for the collision but rather in the formation of a black hole.

Evidently in a theory combining somehow quantum field theory and general relativity sharp localization should be unattainable: if one considers the quantum
nature of particles in general relativity he is forced to use particles of higher and higher energies to reduce the quantum uncertainties, but eventually the energy density at the event one is trying to localize will be high enough to create a black hole, rendering the operative procedure meaningless. The very rough analysis of the last sentence can be performed with more precision, and the resulting limit on particle localization established in this way is exactly the Planck length:

\[ L_P = \sqrt{\frac{G\hbar}{c^3}} \]

which is the length constant obtained from the fundamental constants of quantum mechanics and general relativity. On the other side if we try to consider spacetime 'susceptibility' to matter and energy content in quantum field theory, we are forced to limit the energy of our probe particles in order not to perturb the physical quantities we are trying to measure, having a similar limitation in the outcome precision. From this heuristic analysis we are encouraged to think that considering both quantum mechanical and gravitational aspects of the laws of physics we are forced to renounce to any sharp localization measurement procedure, and thus we are forced to abandon our common notion of classical spacetime. The main content of this thesis will be to analyze some possible deformations classical theories are subjected to, when replacing the notion of classical spacetime with a quantum spacetime.

The first example of quantum spacetime was actually introduced in literature for reasons not directly connected to quantum gravity [5]; it was proposed that in order to cure divergences in quantum field theories one should take into account a possible fuzzy nature of spacetime at the smallest scales. This fuzzy nature was hypothesized to be encoded in an Heisenberg-like uncertainty principle, which would apply however only to spacetime coordinates. The uncertainty principle was based on non-trivial commutation relations between coordinates, which in this particular case assumed the form:

\[ [x^\mu, x^\nu] = i\ell^2 M_{\mu
u} \]

where \( \ell \) was the length scale at which such uncertainty would become noticeable, and \( M_{\mu
u} \) are the standard Lorentz algebra generators. Other uncertainty relations where obtained in [6], this time having in mind the quantum gravity problem and the kind of gedanken experiments I proposed before. This was the starting point of the non-commutative Spacetime research line, which assumes in general commutation relations between coordinates of the form:

\[ [x^\mu, x^\nu] = i\Upsilon^{\mu\nu}(p, x) \]

which of course imply a series of uncertainty principles in position measurements:

\[ \delta x^\mu \delta x^\nu \geq \frac{1}{2} |\langle \Upsilon^{\mu\nu}(p, x) \rangle| \]

A very important peculiarity of these deformed commutation rules is the behaviour of the matrix \( \Upsilon^{\mu\nu} \) under the action of the symmetry group. There are two main choices: if we treat \( \Upsilon \) like a normal tensor under the Lorentz transformation group, we would not have any non-trivial deformation of the symmetry structure of the
theory, but we could use the commutators between spacetime coordinate to select a preferred reference frame (given that the expectation values of the matrix elements $\Upsilon^{\mu\nu}$ will be different in different reference frames). I want to avoid a breaking of symmetry of the theory, i.e. I want to preserve a 10-parameters symmetry group. Not to lose the principle of (special) relativity, however, as we will see the symmetry group will have to be deformed, in order to accommodate for the $\Upsilon^{\mu\nu}$ to be constant, not changing from one reference frame to the other.

In this thesis I will be mostly interested in three forms of the matrix $\Upsilon^{\mu\nu}$; I will treat the so called $\kappa$-Minkowski spacetime:

$$[x^i, x^0] = i\ell x^i \quad [x^i, x^j] = 0$$

which has a simple Lie-algebra structure and preserves the rotational sector of the Lorentz group (the boost sector has to be deformed to allow for the length scale $\ell$ to be an invariant instead of undergoing the standard Lorentz contraction).

The second example studied will be the already introduced Snyder spacetime:

$$[x^\mu, x^\nu] = i\ell^2 M^{\mu\nu}$$

which, given the covariant form of the commutation relations, retain the full Lorentz symmetry algebra (but requires a novel description of translation transformations).

The third example - that I will however treat only marginally - is the so called Canonical Spacetime:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}$$

where $\theta$ is a constant matrix, invariant under the deformed Lorentz group, which elements have dimension of length and are assumed of the order of $L_P$. The three non-commutative spacetimes shown above contemplate the presence of an invariant scale - the Planck length/energy - in a fully relativistic way: every observer will have the commutation relations between coordinates in exactly the same form, in a similar way in which the speed of light assumes the same numerical value in every boosted frame. In special relativity one had to deform the Galilean group of symmetries in the Poincaré one; here one has to deform the Poincaré group in something more general, that in the two cases of $\kappa$-Minkowski and Canonical spacetimes will be in the category of the so called Hopf Algebras.

Another ingredient of paramount importance in this thesis will be the introduction of a covariant formalism of quantum mechanics, proposed some years ago in [7, 8, 9, 10]. One primary issue of non-commutative spacetime theories is in the physical interpretation of variables. In particular in previous approaches to the theory it was not clear how the time coordinate should be interpreted. Assuming coordinates to be non-commutative objects, we are implying they can be represented as self-adjoint operators on some Hilbert space, and that these commutators are not commutative when $\ell \neq 0$. With this premise we expect in the limit $\ell \to 0$ the theory to contain four coordinate operators with trivial commutation rules between themselves. I will stress this important point once again: the defining commutators shown above treat space and time coordinates in a symmetric way, so we have the expectation that also in the limit $\ell \to 0$ this symmetry remains.

We have however no commutative theory in which time is an operator on the same ground as space coordinates. So non-commutative spacetime theories cannot
be considered a deformation of any known theory of this kind, at least not in the
standard formalism. The usual approach to this issue has been to consider quantum
coordinates only through fields; in field theories coordinates are indeed treated in a
symmetric way: both space and time are considered trivial labels, to identify fields
values. So the idea was to build fields on the quantum coordinates, which would
take operators values just because of the nature of coordinates, not because of the
usual second-quantization non-commutativity. This approach however has some
interpreational and conceptual difficulties, as we will see in the central part of the
thesis.

The idea to overcome these troubles came from a recent formulation of quantum
mechanics in an symmetric way between space and time coordinates. It was shown
in [7, 10] that time can be considered as an operator on the same ground as space
coordinates. This new formulation gave us the baseline of a space/time symmetric
theory, that we could deform to accomodate the non-trivial commutation rule we
want to represent.

The consequence of this representation is that dynamics of particles on a non-
commutative spacetime can be seen as a deformation of standard (relativistic, if we
want a deformation of Minkowski space) quantum mechanics. I am in the position
to predict non-commutative corrections to every quantum mechanical observable
regarding propagation of free particles (being the theory a deformation of first-
quantized relativistic quantum mechanics, I cannot treat interactions). As we
will see, this representation of non-commutative spacetimes provides a much more
compelling picture - both in terms of interpretation and physical results - then the
previous, field theory based analysis.

The outline of the thesis will be the following: the first part is directed to an
overview of technical notions necessary to introduce non-commutative spacetimes
and their representations. In the first chapter I will introduce the notion of Hopf
algebras, crucial in the description of quantum spacetime symmetries, starting from
basic concepts like associative and Lie algebras. The second chapter is dedicated to
the covariant formalism of quantum mechanics: starting from the usual formalism
of galilean relativistic quantum mechanics I will show how the same results can
be derived in a more general formalism, in which time/space asymmetry is not an
a-priori given in the theory, but more a result of the dynamics we want to describe.
As a proof of the flexibility of the theory I will show how it is easily adapted to
describe special relativistic quantum mechanics.

The second part of the thesis is focused on an account of previous approaches to
non-commutative spacetime. In particular in the third chapter I will characterize the
$\kappa$-Minkowski spacetime and its full symmetry structure. In the fourth chapter
the field-theoretic approach is unfolded, and in this context I will prove a number of
results about transformation parameters which can give an overview of the reasons
why the covariant quantum mechanics representation of non-commutative coordinates
is more satisfactory than the field theoretic one. It is shown that to have a compelling
Noether charge structure transformation parameters cannot be accomodated in a
representation of spacetime coordinates alone, but instead a representation in phase
space is required.

This kind of representations is exactly the ones I will describe in the last part
of the thesis. In the last chapters I will move from field theories defined on non-
commutative coordinates, and from their limitations, to the theory of quantum particles on a non-commutative spacetime. In such a way it will be clear how this formalism is useful in non-commutative coordinates representation, and why it can give a much clearer physical account of the quantum features of spacetime. Chapter five illustrates how the covariant quantum mechanics formalism can be deformed to accomodate a non trivial spacetime structure. In chapter six $\kappa$-Minkowski spacetime is reformulated in this new setting, and with the new tools acquired I derive new physical effects which were not accessible in previous treatments. The concept of quantum relative locality will be introduced, a generalization of a - till now - classical effect linked to momentum space curvature [11], to the quantum regime. The last chapter is focused on Snyder spacetime, the oldest example of non-commutative spacetime which, looked from our new perspective, will reveal an aspect completely new with respect to the one described in previous literature with more heuristic approaches.
Part I

Technical preliminaries
Chapter 1

Hopf Algebras

In this section I want to formulate the idea of symmetry of a non-commutative spacetime in a rigorous mathematical language: I start from the formal definition of abstract algebra, using the language of commutative diagrams to give a grasp of the main features of these structures. Then I characterize Lie algebras and universal enveloping algebras, crucial concepts in the study of symmetry in physics. After that I will make the crucial step of introducing the concept of coalgebra, that will guide us to the introduction in the end of the chapter of Hopf algebras, mathematical structures capable of formalizing consistently all the properties needed in the symmetry structure of the non-commutative space-times which I shall consider. In this chapter I will mainly follow the exposition given in [12, 13, 14].

1.1 Algebras

Our basic mathematical objects are of a rather general (and largely studied) type: they belong to the category of algebras. By an (associative) algebra \((\mathcal{A}, +, \cdot, \mathbb{K})\) I mean a set \(\mathcal{A}\) endowed with two operations \((\cdot, +)\) which give it a(n associative) ring structure, and an action of the field \(\mathbb{K}\) (which I can always consider the field of complex numbers) compatible with the two operations.

More explicitly, the two operations and the action of \(\mathbb{K}\) must observe the following axioms:

1. \(a + b \in \mathcal{A}\);
2. \(a + b = b + a\) (commutativity of the sum);
3. \((a + b) + c = a + (b + c)\) (associativity of the sum);
4. \(\exists 0 \in \mathcal{A}\) such that \(a + 0 = a\);
5. \(\exists -a \in \mathcal{A}\) such that \(a + (-a) = 0\);
6. \(a \cdot b \in \mathcal{A}\);
7. \(a \cdot (b \cdot c) = (a \cdot b) \cdot c\) (associativity of the product);
8. \(a \cdot (b + c) = a \cdot b + a \cdot c\) and \((b + c) \cdot a = b \cdot a + c \cdot a\) (distributivity of the product);
9. \( \alpha a \in \mathcal{A} \);

10. \( \alpha(a \cdot b) = (\alpha a) \cdot b = a \cdot (\alpha b) \) (compatibility of the action by \( \mathbb{K} \)).

for every \( a, b, c \in \mathcal{A}, \alpha \in \mathbb{K} \).

These axioms simply tell us that \((\mathcal{A}, +)\) is a vector space over \( \mathbb{K} \), endowed with a compatible product. To remember more easily these axioms we can depict them as commutative diagrams:

\[
\begin{CD}
A \otimes A \otimes A @>{\cdot}>> A \otimes A \\
@VV{\cdot}V @VV{\cdot}V \\
A @>{1}>> A
\end{CD}
\]

This diagram ensures the associativity of the product, being \( \cdot \circ (\cdot \otimes 1) = \cdot \circ (1 \otimes \cdot) \) (and \( \circ \) the composition of maps). I will work with unital algebras, i.e. ones in which exists a unit element \( 1 \) for the product \( \cdot \) in \( \mathcal{A} \). I call \( \eta \) the map that associates with an element \( \alpha \in \mathbb{K} \) the element \( \alpha 1 \in \mathcal{A} \). In diagrams:

\[
\begin{CD}
A \otimes A @>{\eta \otimes 1}>> A \\
@VV{1}V @VV{1}V \\
K \otimes A @= A
\end{CD}
\]

\[
\begin{CD}
A \otimes A @>{1 \otimes \eta}>> A \\
@VV{1}V @VV{1}V \\
A \otimes \mathbb{K} @= A
\end{CD}
\]

1.1.1 Lie algebras

In the first section I explained what an associative algebra is. To the physicist, tough, the word “algebra” is more in touch with the family of generators of some group of transformations than with the formal definition of algebra. In this section I adapt the previous definition to that of Lie algebra, which is precisely the algebra formed by the generators of a Lie group. In abstract terms, there is only one feature that distinguishes Lie algebras from the ones encompassed in our previous definition: the particular choice of the product.

A **Lie algebra** \( \mathcal{L} \) is a non-associative algebra over \( \mathbb{K} \) with product (Lie brackets)
\[ \{,\} : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L} \quad (1.4) \]

Such that:

1. \( \{\alpha a + \beta b, c\} = \alpha \{a, c\} + \beta \{b, c\} \) (linearity);

2. \( \{a, b\} = -\{b, a\} \) (antisymmetry);

3. \( \{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0 \) (Jacobi identity).

For every \( a, b, c \in \mathcal{L} \) and \( \alpha, \beta \in \mathbb{K} \).

As I said an algebra can be seen as a vector space over \( \mathbb{K} \), so, if it has finite dimension, I can choose a basis of so-called generators by which every other element can be obtained with a linear combination.

We can completely specify a Lie algebra by giving the product of its generators, because all other brackets can be obtained by linearity:

\[ \{v, w\} = \{v^i g_i, w^j g_j\} = \sum_{ij} v^i w^j f_{ij}^k g_k \quad (1.5) \]

On the other hand, all the information about products of generators is expressed in the so-called *structure constants*, which specify how much the bracket of two generators is “projected” on the others:

\[ \{g_i, g_j\} = f_{ij}^k g_k \quad (1.6) \]

This expression summarizes the fact that the bracket of two generators is in the algebra, so we can write it as a linear combination of generators.

There is a huge mathematical theory about Lie algebras, but here I report only a glimpse of it, necessary for a satisfactory understanding of the following developments.

### 1.1.2 Universal enveloping algebra

As I said the product of an abstract Lie algebra is a non-associative one, but in all cases of physical interest we have a Lie algebra of operators, in which an associative product is defined, and the Lie brackets are recovered in terms of commutators. This is because in physics we work with Lie algebras representations, rather than abstract algebras, and a representation in terms of operators is an inclusion of the Lie algebra in a bigger one, with an associative product.

That of *universal enveloping algebra* is a concept connected with this inclusion, and tells us what is the most general associative algebra in which a Lie algebra can be embedded recovering the brackets in terms of commutators.

Let’s analyze more closely the relations between Lie brackets and commutators: given any associative algebra \( \mathcal{A} \) we can associate to it a Lie algebra in which the Lie brackets are defined to be the commutator of two elements:

\[ \{a, b\} \equiv [a, b] = a \cdot b - b \cdot a \quad \forall a, b \in \mathcal{A} \quad (1.7) \]
Conversely, given a Lie algebra $\mathcal{L}$ we can always associate to it an associative unital algebra by a universal property, namely that it must be homomorphic to all other algebras homomorphic to the Lie one.

More formally, if we indicate with the symbol $\mathfrak{U}_\mathcal{L}$ the algebra in which the original product of $\mathfrak{U}$ is replaced by the commutators, a pair $(\mathfrak{U}, i)$ is called the universal enveloping algebra of $\mathcal{L}$ if the following properties hold:

- $i : \mathcal{L} \to \mathfrak{U}_\mathcal{L}$ is an algebra homomorphism
- for any algebra homomorphism

\[
h : \mathcal{L} \to \mathfrak{U}_\mathcal{L}
\]

we have a third homomorphism:

\[
h' : \mathfrak{U} \to \mathfrak{U}
\]

such that

\[
h = h' \circ i
\]

In terms of commutative diagrams we have:

\[
\begin{array}{c}
\mathfrak{U} = \mathfrak{U}_\mathcal{L} \\
i \\
\mathcal{L} \quad h \quad \mathfrak{U} = \mathfrak{U}_\mathcal{L} \\
\end{array}
\]

We can construct directly the universal enveloping algebra of a given Lie algebra in terms of its tensor product space

\[
\mathcal{T}(\mathcal{L}) = K \oplus \mathcal{L} \oplus (\mathcal{L} \otimes \mathcal{L}) \oplus (\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L}) \oplus ... = \bigoplus_i \mathcal{L}^\otimes_i
\]

the universal enveloping algebra of $\mathcal{L}$ can be recovered as the space

\[
\mathfrak{U} = \mathcal{T}(\mathcal{L})/\mathcal{I},
\]

where $\mathcal{I}$ is the ideal generated by elements of the form

\[
i = a \otimes b - b \otimes a - \{a, b\}, \quad a, b \in \mathcal{L}
\]

So defined, $\mathfrak{U}$ is the only associative algebra satisfying the universal property given above.

\[\text{A (two-sided) ideal } \mathcal{I} \text{ of the algebra } \mathcal{A} \text{ is an additive subgroup of } \mathcal{A} \text{ with the property that for every } h \in \mathcal{I}, h \cdot a \in \mathcal{I}, a \cdot h \in \mathcal{I} \forall a \in \mathcal{A}.\]
1.2 Coalgebras

I can now introduce an abstract structure that is somehow “dual” with respect to that of algebra; this is the coalgebra, and is the starting point to switch to Hopf algebras.

The definition for a coalgebra $\mathcal{C}$ is very similar to that of algebra, apart from the product and the unit element, that here are switched in the coproduct and counit map. The coproduct is a map

$$\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$$  \hspace{1cm} (1.15)

that has to be coassociative:

$$(1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta$$  \hspace{1cm} (1.16)

and compatible with the counit map $^2\epsilon : \mathcal{C} \to K$:

$$(1 \otimes \epsilon) \circ \Delta = (\epsilon \otimes 1) \circ \Delta = 1$$  \hspace{1cm} (1.17)

I can try to make this definitions a little less abstract explaining the relationship between algebras and coalgebras; more precisely I can clear up in what sense they are dual.

The fact is easier to understand when formulated in terms of commutative diagrams; the important ones for the coalgebras are:

\[ \text{coproduct} \]

\[ \text{counity} \]

\[ ^2\text{The coalgebra is, like the algebra, a vector space on a field } K. \]
As we can see the last three diagrams can be obtained from the corresponding algebraic ones reverting the direction of all the arrows. To better understand this fact I analyze it in two fundamental cases, comparing the product and the unity with the corresponding coalgebric elements:

\[ W \otimes W \xrightarrow{\Delta} W \]
\[ W \xrightarrow{\epsilon} K \]
\[ W \xrightarrow{\eta} W \]

That is, the effect of switching from algebras to coalgebras is the change of direction of the arrows of the fundamental maps of product and unit; every commutative diagram built with this operations, thence, has all the directions switched.

### 1.2.1 Sweedler notation

There is an handy and useful notation to indicate coproduct of elements of a coalgebra \( C \): the so-called Sweedler notation. For the most general coproduct of an element we have:

\[ \Delta(c) = \sum_{ij} c^{ij} \otimes ij \]

where \( j_i \) are elements of the basis\(^3\) of \( C \). The Sweedler notation consists in a number of simplifications, that leads us to remove the summation symbol:

\[ \Delta(c) = \sum_i c_{i(1)} \otimes c_{i(2)} = \sum_{(i)} c_{(1)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)} \]

so, whenever we encounter one of the expressions given above we have to understand that the first type of sum is intended (where \( c_{i(1)}, c_{i(2)} \) are generic elements in the first and second space respectively).

With this notation we have:

\[ \sum_{(a)} a_{(1)} \otimes \left( \sum_{(a_{(2)})} a_{(2)(1)} \otimes a_{(2)(2)} \right) = \sum_{(a)} \left( \sum_{(a_{(1)})} a_{(1)(1)} \otimes a_{(1)(2)} \right) \otimes a_{(2)} = a_{(1)} \otimes a_{(2)} \otimes a_{(3)} \]

\(^3\)If we don’t have a basis the sum indexes run over generic elements of \( C \).
\[ \sum_{(a)} a(1) \otimes \epsilon(a(2)) = \sum_{(a)} \epsilon(a(1)) \otimes a(2) = a \] (1.25)

for coassociativity and counit map.

I can show another way in which algebras and coalgebras are dual structures using Sweedler notation; as repeatedly said they are vector spaces, so they admit dual spaces, that I will call \( A^* \) for an algebra \( A \), and \( C^* \) for a coalgebra \( C \). I can define a product in \( C^* \) and a coproduct\(^4\) in \( A^* \), using the primitive coproduct and product in \( C \) and \( A \):

\[
(\phi \cdot \psi)(c) \doteqdot \phi(c(1))\psi(c(2)) \quad \forall c \in C, \phi, \psi \in C^* \quad (1.28)
\]

\[
\Delta(f)(a, b) \doteqdot f(a \cdot b) \quad \forall a, b \in A, f \in A^* \quad (1.29)
\]

and similarly, for unit and counit:

\[
\eta(\ell)(c) \doteqdot \ell c \quad \forall \ell \in \mathbb{K}, c \in C \quad (1.30)
\]

\[
(\epsilon f)(\ell) \doteqdot f(\epsilon(\ell)) \quad \forall f \in A^*, \ell \in \mathbb{K} \quad (1.31)
\]

Another dual concept between algebras and coalgebras is that of commutativity/cocommutativity; defining the map \( \tau : A \otimes A \to A \otimes A \) (here \( A \) can be an algebra or a coalgebra) with action

\[
\tau(a \otimes b) = b \otimes a \quad \forall a, b \in A \quad (1.32)
\]

I can write the commutativity condition for algebras as

\[
\cdot \circ \tau = \cdot \quad (1.33)
\]

and similarly, for coalgebras:

\[
\tau \circ \Delta = \Delta \quad \text{or} \quad \Delta(c) = c(1) \otimes c(2) = c(2) \otimes c(1) \forall c \in C \quad (1.34)
\]

In the last equation it has to be pointed out that \( c(1) \otimes c(2) = c(2) \otimes c(1) \) doesn’t imply that each term \( c(i(1)) \otimes c(i(2)) \) in the summation is equal to \( c(i(2)) \otimes c(i(1)) \).

\(^4\)To be rigorous the second definition is valid only in the finite dimensional case, because the pullback of the map \( \cdot \) maps \( A^* \) in \( (A \otimes A)^* \):\n
\[
\cdot : A^* \to (A \otimes A)^*, \quad (1.26)
\]

and only in the finite dimensional case we can establish what is the isomorphism

\[
(A \otimes A)^* \cong A^* \otimes A^* \quad (1.27)
\]

the fact that to every coalgebra we can associate a dual algebra, but not the converse, is a sign of the fact that coalgebras are more fundamental than algebras themselves.
1.3 Bialgebras and Hopf algebras

The two structures described so far can be merged in a single one, called bialgebra; It is a vector space \((B, +, \Delta, \eta, \epsilon, K)\) over the field \(K\) in which are defined a product and a coproduct (compatible with each other), a unit and a counit map. The result is a space \((B, +, \cdot, \Delta, \eta, \epsilon, K)\), with the compatibility conditions:

\[
\Delta(a \cdot b) = a_{(1)} \cdot b_{(1)} \otimes a_{(2)} \cdot b_{(2)}, \quad \Delta(1) = 1 \otimes 1, \quad \epsilon(a \cdot b) = \epsilon(a) \epsilon(b) \quad (1.35)
\]

for every \(a, b \in B\). Adding to this structure a notion of “inverse” I obtain the goal of this chapter: a Hopf algebra.

A Hopf algebra \((H, +, \cdot, \Delta, \eta, \epsilon, S, K)\) is a bialgebra \((H, +, \cdot, \Delta, \eta, \epsilon, K)\) equipped with a linear antipode map \(S : H \to H\), which have the following compatibility property:

\[
\cdot \circ (1 \otimes S) \circ \Delta = \cdot \circ (S \otimes 1) \circ \Delta = \eta \circ \epsilon \quad (1.36)
\]

or, in Sweedler notation:

\[
h_{(1)} \cdot S(h_{(2)}) = S(h_{(1)}) \cdot h_{(2)} = \eta(\epsilon(h)) \quad (1.37)
\]

As usual I present these axioms also in form of commutative diagrams:

\[
\begin{array}{c}
H \otimes H \xrightarrow{\cdot} H \xrightarrow{\Delta} H \otimes H \\
\downarrow \Delta \otimes \Delta \quad \quad \quad \quad \downarrow \cdot \otimes \cdot \\
H \otimes H \otimes H \otimes H \xrightarrow{1 \otimes \tau \otimes 1} H \otimes H \otimes H \otimes H
\end{array}
\]

\[
\begin{array}{c}
H \otimes H \xrightarrow{\epsilon} K \quad \quad \quad K \xrightarrow{\eta} H \\
\downarrow \epsilon \otimes \epsilon \quad \quad \quad \downarrow \eta \otimes \eta \\
H \otimes H \quad \quad \quad H \otimes H \xrightarrow{\Delta} \quad \quad \quad \quad \quad \quad (1.39)
\end{array}
\]

\[
\begin{array}{c}
H \xrightarrow{\epsilon} K \quad \quad \quad K \xrightarrow{\eta} H \\
\downarrow \Delta \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \quad (1.40)
\end{array}
\]

It can be demonstrated that the antipode map must also respect the following properties:

1. \(S \circ \cdot = \cdot \circ (S \otimes S)\)
2. \(S(1) = 1\)
3. \((S \otimes S) \circ \Delta = \tau \circ \Delta \circ S\)

4. \(\epsilon \circ S = \epsilon\)

Looking carefully all this commutative diagrams we can realize that they have an interesting property: they are sent into themselves in reversing the arrows; this is another manifestation of the algebra-coalgebra duality. Being the Hopf algebras both algebras and coalgebras, they are in this sense self-dual structure.

To exhibit an example of Hopf algebra we can equip with the necessary condition a universal enveloping algebra:

\[
\Delta(g_i) = g_i \otimes 1 + 1 \otimes g_i \quad (1.41)
\]

\[
\epsilon(g_i) = 0 \quad \forall i, \quad \epsilon(1) = 1 \quad (1.42)
\]

\[
\Delta(1) = 1 \otimes 1, \quad \eta(\ell) = \ell 1 \quad \forall \ell \in K \quad (1.43)
\]

The compatibility conditions (1.35) are automatically satisfied between the coproduct so defined and the Lie brackets:

\[
\Delta(g_i \cdot g_j) = \Delta(f^k_{ij} g_k) = f^k_{ij}(g_k \otimes 1 + 1 \otimes g_k) = (g_i \otimes 1) \cdot (g_j \otimes 1) + (1 \otimes g_i) \cdot (1 \otimes g_j) \quad (1.44)
\]

The antipode map required to obtain a Hopf algebra is defined as:

\[
S(g_i) = -g_i \quad S(1) = 1 \quad (1.45)
\]

1.4 Dual Hopf Algebras

As I said every coalgebra defines an algebra structure on its dual space, and under some conditions also every algebra defines a coalgebra on its dual space. From this propositions we deduce that every Hopf algebra, being at the same time an algebra and a coalgebra, defines another Hopf algebra as its dual space.\(^5\)

Writing the action of the dual \(H^*\) to an Hopf algebra \(H\) as:

\[
< \phi, v > = \phi(v) \quad \forall \phi \in H^*, h \in H, \quad (1.46)
\]

the condition for two Hopf algebras \(H_1\) and \(H_2\) to be a dual pair is:

\[
< \Delta(v), w \otimes z > = < v, w \cdot z > \quad (1.47)
\]

\[
< v \cdot u, w > = < v \otimes u, \Delta(w) > \quad (1.48)
\]

\[
< 1, w > = \epsilon(w) \quad (1.49)
\]

\[
< v, 1 > = \epsilon(v) \quad (1.50)
\]

\(^5\)I said that Hopf algebras are autodual as a structure, not that a single Hopf algebra is its own dual.
for all \(v, u \in H_1, w, z \in H_2\). Furthermore we have a condition on the antipode map:

\[
\langle S(v), w \rangle = \langle v, S(w) \rangle \tag{1.51}
\]

With these definitions we could easily prove that if \(H_1\) is a Hopf algebra the same is true for \(H_2\), with coproduct, counit and antipode defined as above. As I said in the algebra/coalgebra case, the definitions given above can be used to define the dual coalgebra of an algebra only in the finite dimensional case, but they can be used also in the infinite dimensional one to establish, given two Hopf algebras, if they are or not a dual pair. When the relation \(\langle \cdot, \cdot \rangle\) between \(H_1\) and \(H_2\) is nondegenerate we say that the two Hopf algebras are a *strictly dual pair*.

As we will see later that of duality is the condition that links the algebras of coordinates and translation transformations:

\[
\langle P_\mu, x^\nu \rangle = -i \delta^\nu_\mu \tag{1.52}
\]
Chapter 2

Covariant quantum mechanics

Quantum mechanics is at the foundation of our understanding of the world. It is one of the most experimentally successful physical theories we have, and in its galilean relativistic formulation also its mathematical grounds are perfectly solid. Problems start to arise when we want to explore the special relativistic regime, in which it is a well known fact that a more accurate description of reality is given by field theories, not particle theories. There is however a grey zone - in particular free theories - in which predictions of special relativistic quantum mechanics can still be considered reliable. In the standard, flat commutative spacetime setting this restriction is not satisfactory, in that we can study only particle propagation, and it is a quite trivial problem. However in our interest, the study of non-commutative spacetime, the effects of a quantum spacetime could be nontrivial also on particle propagation, and so the situation is worth studying, as we will see.

The standard quantum mechanics formalism is however too rigid to be adapted to a non-commutative spacetime: in our Hilbert space we don’t have any room for a time observable. To deform the classical theory in a non-commutative one we need a more covariant formulation of quantum mechanics. This covariant formalism was introduced by other authors \[7, 8, 9, 10\] for different purpose, but can nonetheless be perfectly adapted to our scopes. The main peculiarity of this formalism, that I will soon introduce, is the splitting of states and observables in "kinematical" and 'physical’ ones, the distinction being between what we just observe (basically any physical quantity) and what we can predict (only correlations between measurable quantities). I will give a short review of this formalism, but for a more complete exposition one can read \[3\].

2.1 Galilean quantum mechanics

In this section I will review Galilean quantum mechanics, emphasizing the aspects that will be changed in the covariant formalism. The main ingredients of the standard formulation of quantum mechanics are:

- an Hilbert space of states \(H_0\); in the usual, 3-dimensional flat spacetime this Hilbert space is \(L^2(\mathbb{R}^3, d^3p)\), the space of square integrable functions of three variables, with scalar product given by the standard Lebesgue integration. In
the following I will always work in the momentum representation, which is the most practical in our setting.

- Observables $\mathcal{O}$, which are self-adjoint operators on $\mathcal{H}_0$. The spectrum of the operator $\mathcal{O}$ is the set of possible results for any measurements of the associated observable. Moreover, in the usual measurement description, once the observable $\mathcal{O}$ has been measured to have a given eigenvalue $\lambda$ the system is assumed to be in the associated eigenspace.

- Time evolution can be described by evolution of states, while observables are time independent (Schrödinger equation):

$$i\frac{\partial \psi}{\partial t} = H_0\psi$$

or evolution of observables, while states are time independent (Heisenberg equation):

$$i\frac{\partial \mathcal{O}}{\partial t} = [\mathcal{O}, H_0]$$

- Probabilistic interpretation of the system state: once we know the system is in a state $\psi$ the only prediction we can make about the measurement of a physical quantity $\mathcal{O}$ is the probability distribution:

$$P(\lambda) = |\langle \lambda | \psi \rangle|^2$$

with $|\lambda\rangle$ eigenstate of the observable $\mathcal{O}$ with eigenvalue $\lambda$.

From the above given scheme we can conclude that a well defined quantum mechanics for a single particle is given with an Hilbert space $\mathcal{H}_0$ and a time evolution $H_0$.

As already pointed out in the usual flat, 4-dimensional spacetime with galilean symmetry, the Hilbert space $\mathcal{H}_0$ is the space of the square integrable functions in three variables, and the operator $H_0$ is the galilean hamiltonian of the usually written in the form:

$$H_0 \left( p_i, -i \frac{\partial}{\partial p_i} \right) = \frac{p_i^2}{2m} + V \left( -i \frac{\partial}{\partial p_i} \right)$$

In this setting we have three position observables $x^i$ and three momentum observables $p_j$, plus the hamiltonian $H_0$, but not a time observables. The three spatial position and momentum operators comply with canonical commutation relations:

$$[p_i, x^j] = i\delta_i^j$$

It is evident a strong asymmetry between time and space variables of the theory, that in this form is unsuitable for a relativistic symmetry extension. It is possible to show that this asymmetry is only a feature of the dynamics we choose to describe with our theory, and it is not implied by the formalism of quantum mechanics.

\footnote{I am assuming the states to be already normalized.}
I can obtain the same theory starting with a more general, more symmetric space, containing auxiliary variables, and then choosing a constraint limiting the number of degrees of freedom\(^2\).

The starting point is called *kinematical* Hilbert space, and for example to reproduce the standard galilean quantum mechanics in \(3 + 1\) dimensions this is the space of square integrable functions on \(\mathbb{R}^4\), with flat integration measure. In this space I can define 4 multiplicative operators \(p_\mu\), and 4 conjugate variables \(q^\nu\) having canonical commutation relations:

\[
[p_\mu, q^\nu] = i\delta^\nu_\mu
\]  

So I have now the four components of momentum on the same ground, and a fourth coordinate operator \(q^0 = -i\frac{\partial}{\partial p^0}\) re-establishing the symmetry between space and time coordinates.

Though for a complete and satisfactory account of this new formalism the reader will be forced the reference, I will for comparison with the standard formalism give the new axioms of this covariant theory, and then proceed to show that in case of galilean symmetry the same results as the classical theory are obtained. In the next section I will show how easy is to adapt this covariant formalism to more general symmetry groups.

This covariant theory is based on the following:

- an auxiliary Hilbert space, called *Kinematical* Hilbert space \(\mathcal{K}\); in the usual flat, 4-dimensional spacetime, it is the space of square integrable functions in *four* variables:
  \[
  \mathcal{K} = L^2(\mathbb{R}^4, d^4p)
  \]  
  This space cannot describe any particle dynamics, it has only information about empty spacetime. The states in it can be approximately seen as 'probability distributions of events'.

- Hermitian operators in this space are called *partial observables*, although complying with the standard definition of observables, they are not associated to any particle dynamics, so they are not physical, but contain gauge (not physical) degrees of freedom. Partial observables are for example the coordinates we will deform to obtain the non-commutative commutation relations; in fact one particle dynamics is introduced, we will see that it is not possible to have four commuting coordinates which are the components of a Lorentz vector (all features we want to have for our coordinates in the commutative limit). The commutative kinematical coordinates are:
  \[
  q^\mu = -i\frac{\partial}{\partial p_\mu}
  \]  
  Instead of a time evolution we will have a *constraint* \(H\) to remove the non-physical degrees of freedom (this formalism is a quantum analogue of the constrained formulation of classical mechanics). There is an insightful way to

\(^2\)A similar reformulation can be given of classical mechanics, see for example [15].
see the need for a constraint on the kinematical space. Hermitian operators on the kinematical space are all the physical quantities one can measure: readings of clocks, positions of particles.

Dynamic quantities, however, are observables that we predict. Even though we are able to measure the position of a particle, the results of the measure are meaningless, unless we correlate them with - for example - the reading of a clock. Indeed the only way we can tell the result of a measure is related to a particle is to correlate that measure outcome to other similar observables (think to the trajectory of a particle, \(x(t)\)). Correlations are the only quantities the theory can predict, so they are the only results of our dynamics. To get rid of other, pure gauge degrees of freedom, we impose our states to respect a constraint:

\[ H\psi = 0 \]  

which will substitute also the standard hamiltonian time evolution. For example, in galilean quantum mechanics, we consider as physical only states which are solutions of the Schrödinger equation, so the constraint will be:

\[ H\psi(p_\mu) = \left[ p_0 - H_0 \left( p_i, -i\frac{\partial}{\partial p_i} \right) \right] \psi(p_\mu) = 0 \]  

(in the kinematical states we have function of four independent components of the momentum, so this constraint amounts to having particles with physical energies). We will see how the constraint can be generalized to accommodate more general symmetry structures.

Another way to see this formalism is that - given we are looking for a covariant theory - this is a theory of worldlines, not of spatial probability distribution like quantum mechanics in the standard setting. Worldline states, i.e. states complying with the constraint, will be called Physical states.

In [3] a "projector" operator is introduced, which associate to any state in the kinematical space a well defined physical state, complying with the constraint\(^3\). The form of this projector is:

\[ P = \int e^{irH} \, d\tau \sim \delta(H) \]  

• We can now introduce a second Hilbert space, the Physical Hilbert space, or the space of physical states. As we will see the physical states are not elements of the kinematical Hilbert space. After the imposition of the constraint their Fouries transform does not depend on \(p_0\) anymore, so they are unbounded states in the conjugate variable \(q^0\) (of course, we expect a worldline to be an unbounded function in the time direction).

To obtain a well defined Hilbert space structure for the physical states we have to introduce a new scalar product, with respect to which physical states are normalizable. The physical scalar product can be put in the form:

\[ \langle \psi | \phi \rangle \equiv \int \bar{\psi}(p)\phi(p)\delta(H) \, d^4p \]  

\(^3\)It is not a full fledged projector, in that as we will see it is a many-to-one map but states in its image are not contained in its domain.
Here $\psi$ and $\phi$ are members of equivalence classes representing the respective physical states through the projector.

This definition seems to rely on the kinematical states chosen, but the presence of the delta in the integration measure make that dependence void. Using this scalar product we can consider different states only different congruence classes of kinematical states, where the congruence relation is given by whether or not they are mapped in the same physical state.

Considering the norm completion of this space with the new scalar product we have a new Hilbert space, that we will call physical Hilbert space. A consequence of the utmost importance of the introduction of the new scalar product is that hermitian operators in the kinematical space will not be, in general, hermitian operators in the physical space. So we have to be careful not to mistake observables relative to particles (so living in the physical space) with kinematical observables, which have nothing to do with particles.

It is easy to see that a necessary condition for a kinematical hermitian operator $O$ to be hermitian also in the physical space is the commutativity between $O$ and the hamiltonian constraint $H$. If we want to describe free particles this is the case, for example, of the momentum components $p_i$ or every function of them, but not of the kinematical coordinates $q^\mu = -i\frac{\partial}{\partial p^\mu}$. The particle coordinates introduced in the standard formalism of quantum mechanics are then not the kinematical coordinates $q^i$, but a function of them commuting with the hamiltonian constraint. There exist a one-parameter family of operator having the right properties under the galilean symmetry group, and the parameter plays exactly the role we assign to time in the standard formulation of quantum mechanics:

$$A^i(T) \equiv q^i - \frac{[H, q^i]}{[H, q^0]} q^0 + \frac{[H, q^i]}{[H, q^0]} T + h.c. \quad (2.13)$$

$$A^0(T) \equiv T, \quad T \in \mathbb{R} \quad (2.14)$$

For free, galilean-relativistic particles this becomes:

$$A^i(T) \equiv q^i - \frac{p^i}{m} q^0 + \frac{p^i}{m} T \quad (2.15)$$

$$A^0(T) \equiv T, \quad T \in \mathbb{R} \quad (2.16)$$

- As the last ingredient of the theory we have the probabilistic interpretation. This part of the theory is very similar to the usual formalism, but now we have a different scalar product. We can still understand the function $|\langle \lambda | \psi \rangle|^2$ as the probability distribution to have outcome $\lambda$ from a given measurement if $|\lambda\rangle$ is an eigenstate of the observable $O$ we are measuring.

We can however perform another type of predictions - or, better, put predictions in a different form. Considering the states in their Fourier representation the physical scalar product can be put in the spacetime form:

$$\langle \phi | \psi \rangle = \int \overline{\psi}(p)\phi(p)\delta(H) \, d^4p = \int \overline{\psi}(x)W(x - y)\phi(y) \, d^4x \, d^4y \quad (2.17)$$
where $W(x - y)$ is the propagator of the theory\footnote{We will see in the special relativistic theory that the form of the propagator can be a little more general than this.}:

$$W(x - y) = \int d^4p \delta(H)e^{ip(x-y)} \quad (2.18)$$

With the scalar product in this form it is easy to see that we can interpret the wave function $|\langle \psi | \phi \rangle|^2$ as the probability that the worldline $W \phi$ will cross the spacetime region $\psi$ (this statement needs more precision, see \cite{3} for a satisfying account on this interpretation).

To taste the novelties of this formulation I will show how the constraint act on a class of states in this space that we will use much, the gaussians:

$$\psi_{\sigma \mu}(p) = \frac{1}{\pi \sqrt{\prod \sigma_{\mu}}} e^{-\frac{(p_0 - \pi_0)^2}{2\sigma_0^2}} e^{-\frac{(p_i - \pi_i)^2}{2\sigma_i^2}} e^{ip_{\mu}\pi^\mu} \quad (2.19)$$

Considering the Fourier transform of these (kinematical) states:

$$\psi_{\sigma \mu}(q) = \frac{\sqrt{\prod \sigma_{\mu}}}{\pi} e^{-\frac{\sigma_0^2(q_0 - \pi_0)^2}{2}} e^{-\frac{\sigma_i^2(q_i - \pi_i)^2}{2}} e^{iq_{\mu}\pi^\mu} \quad (2.20)$$

we can see that these states describe regions of spacetime, fast decreasing in both space and time direction. Loosely speaking the scalar product on this space evaluate superposition of spacetime regions\footnote{To be more precise, it evaluates superposition on phase space.}, so it is useful to describe coincidences of spacetime events, not to describe particles. In particular it is evident that up to now we don’t have any dynamics; to recover the concept of a time evolution we need the constraint, specifying the relation between the time and space variables, or equivalently between $p_0$ and $p_i$.

Given we want to reproduce galilean quantum mechanics, we already know what is the dynamics: we have just to express Schrödinger equation in the form of a constraint. The states complying with such a constraint will respect the right evolution in time to describe physical particles.

The constraint decides the dynamics of the theory, so it has to be invariant under the symmetry group we choose to describe our physics. As it will be evident in more general settings, the constraint has to be the Casimir of the symmetry algebra, so we have a similar elegant connection between symmetry and dynamics that we find in quantum field theory.

To see the effects of this projector on a spacetime gaussian we work with the Fourier components:

$$\psi_{\sigma \mu}(q) = \frac{\sqrt{\prod \sigma_{\mu}}}{\pi} e^{-\frac{\sigma_0^2(q_0 - \pi_0)^2}{2}} e^{-\frac{\sigma_i^2(q_i - \pi_i)^2}{2}} e^{iq_{\mu}\pi^\mu}$$

$$= \frac{1}{\pi \sqrt{\prod \sigma_{\mu}}} \int e^{-\frac{(\pi_0 - p_0)^2}{2\sigma_0^2}} e^{-\frac{(\pi_i - p_i)^2}{2\sigma_i^2}} e^{ip_{\mu}(\pi^\mu - q^\mu)} d^4p \quad (2.21)$$
If the Hamiltonian describes a free particle it is easy to study the effect of the constraint inserting the dirac delta of \( H \) in the momentum integral. In this way the operator \( H \) becomes a function, and the Dirac delta is well defined:

\[
\delta(H)\psi_{\sigma_\mu}(q) = \frac{1}{\pi \sqrt{\Pi \sigma_\mu}} \int \delta(p_0 - H_0(p_)) \left( e^{-\frac{(p_0 - \tilde{\tau}_0)^2}{2\sigma_0^2}} - e^{-\frac{(p_0 - \tilde{\tau}_0)^2}{2\sigma_0^2}} e^{-ip_\mu(q_\mu - \tilde{\tau}_\mu)} d^3p \right)
\]

\[
= \frac{1}{\pi \sqrt{\Pi \sigma_\mu}} \int e^{-\frac{(H_0(p_\mu) - \tilde{\tau}_0)^2}{2\sigma_0^2}} e^{-\frac{(p_\mu - \tilde{\tau}_\mu)^2}{2\sigma_\mu}} e^{-i(H_0(p_\mu)(q_0 - \tilde{\tau}_0) - p_\mu(q_\mu - \tilde{\tau}_\mu))} d^3p
\]

If we now assume \( \sigma_\mu \ll \sigma_0 \) (time localization much better than space localization, as typical of galilean approximation) we can expand the integrand to obtain again a gaussian:

\[
\delta(H)\psi_{\sigma_\mu}(q) \approx e^{-i\left(H_0(\tilde{\tau}_\mu) - \frac{\partial H_0}{\partial p_\mu}\right)(q_\mu - \tilde{\tau}_\mu)} \int \frac{1}{\pi \sqrt{\Pi \sigma_\mu}} \left( e^{-\frac{(p_\mu - \tilde{\tau}_\mu)^2}{2\sigma_\mu}} e^{-\frac{(p_\mu - \tilde{\tau}_\mu)^2}{2\sigma_\mu}} \frac{e^{-\frac{(p_0 - \tilde{\tau}_0)^2}{2\sigma_0^2}}}{2m^2\delta_0} \right) \times \frac{e^{ip_\mu\left((q_\mu - \tilde{\tau}_\mu) - \frac{\partial H_0}{\partial q_\mu}(q_0 - \tilde{\tau}_0)\right)}}{d^3p}
\]

Applying the galilean approximation and getting rid of useless phases we find that the state in spacetime representation is a gaussian centered around the classical worldline \((q_i - \tilde{\tau}_i) - \frac{\partial H_0}{\partial q_i}(q^0 - \tilde{\tau}^0) = 0\), with uncertainty growing along the worldline:

\[
\sigma_{\text{eff}}^2 \approx \frac{1}{\sigma_i^2 + \frac{\sigma^2_i(q_0 - \tilde{\tau}_0)^2}{m^2}}
\] (2.22)

This expression for the state in terms of kinematical variables gives us a pictorial representation of the spacetime region occupied by the particle along its motion, but they cannot be treated like states in the Hilbert space previously defined, in that they are not normalizable (they are not decreasing in the time direction). To work with physical states we have to consider them as elements of the physical Hilbert space, so evaluate them with the physical scalar product.

We want to show now how to recover familiar concepts of galilean quantum mechanics, like for example the probability distribution in space \( |\psi(x)|^2 \). With the physical scalar product we have seen that an interpretation of probability distribution in spacetime can be obtained in an easy way.

We want to evaluate the probability to find a particle on the worldline specified by \( \psi \) in the infinitesimal spacetime region \( \langle y|R \rangle = \delta^3(y - x)\delta(y_0 - x_0) \).

As a first step we want to verify that the generalized state \( |R\rangle \) is indeed an eigenstate \( |x\rangle \) of the physical position operator. In momentum space it reads:

\[
R(p) = \langle p|R \rangle = e^{-ip\hat{x}} e^{ip^0x^0}
\] (2.23)

and acting on this function with the coordinate operator \( \hat{x} \) for \( T = x^0 \) we indeed obtain:

\[
\hat{A}(x^0)R(p) = -i \left( \frac{\hat{p}}{m} + \frac{\hat{\vec{p}}}{m} \right) \hat{x} R(p) = \hat{x} R(p)
\] (2.24)
so, like in the usual formalism, we have the dirac deltas as eigenstates of the physical position operators (we stress out again that they are very different, both conceptually and in form, from the kinematical ones).

As a next step we have to prove that the physical scalar product give the same result as the one we are used to. The propagator in momentum space, for the galilean theory, is:

$$W(x - y) = \int d^4p \delta(p_0 - H_0) e^{ip(x-y)} = \int e^{-i\vec{p} \cdot (\vec{x} - \vec{y})} e^{iH_0(x_0 - y_0)} d^3p$$  \hspace{1cm} (2.25)

It is easy to note that this propagator, for $x_0 = y_0$, is just a dirac delta, so if the position eigenstate $|x\rangle$ and the state we are measuring $|\psi\rangle$ are localized on the same time slice, the probability density for the position of the particle is the same as in the standard formalism. We have however the possibility to answer with no additional efforts to the same question when the state $|\psi\rangle$ is not localized on the same time slice of spacetime, or when it is not time localized at all. This is the great advantage of a covariant formalism, and it makes clear that the time/space asymmetry is only dynamical generated, in that the propagator is a-priori symmetric in the spacetime variables.

2.2 Special relativistic quantum mechanics

We saw in the previous section how to recover the standard galilean quantum mechanics in the more general, more covariant setting we introduced. In that case it could seem a little overkill, but the power of the covariant formalism comes out when we change the symmetry structure assumed to be valid for the theory. Here we show how we can get, in the same general setting, special relativistic quantum mechanics in an effortless way.

The kinematical space is of the same form as in the galilean case:

$$K = L^2(\mathbb{R}^4, d^4p)$$  \hspace{1cm} (2.26)

Now however the 4 variables $p_\mu$ are seen as components of a four-vector under Lorentz transformations, not just galilean ones:

$$p'_\mu = \Lambda^\nu_\mu p_\nu \hspace{1cm} , \hspace{1cm} \Lambda^\nu_\mu \Lambda^\sigma_\rho \eta_{\nu\sigma} = \eta_{\mu\rho}$$  \hspace{1cm} (2.27)

and the same is valid for the conjugate variables $q^\mu$.

As already stated in the previous sections, the constraint has to be invariant under the symmetry algebra, and moreover we want it to reduce to the Schrödinger equation in the galilean limit. It is obvious than that we have to take as a constraint the mass casimir of the Lorentz group:

$$H = p_0^2 - p_i^2 - m^2$$  \hspace{1cm} (2.28)

To implement the correct physical scalar product we use the same prescription, we deform the integration measure introducing the projector. In this case we are somewhat forced to consider just free particles, in that we know an interacting relativistic quantum mechanics is not a sensible theory of nature. In the special
relativistic regime, in addition to the galilean case we add to the projector the prescription of positiveness of energy, though we could avoid it and retain also negative frequencies components in our physical states. This choice ultimately amounts to the kind of propagator we want for our theory. We stick again to the momentum space representation:

\[
(\psi | \phi) \equiv \int \bar{\psi}(p) \phi(p) \delta (H) \Theta(p_0) \, d^4p = \\
= \int \bar{\psi}(p) \phi(p) \delta \left( p_0^2 - p_i^2 - m^2 \right) \Theta(p_0) \, d^4p = \\
= \int \frac{1}{2p_0} \bar{\psi}(\tilde{p}) \phi(\tilde{p}) \, d^3p
\]

where in the last line we have introduced the physical momentum:

\[
\tilde{p}_i = p_i, \quad \tilde{p}_0 = E(\tilde{p}) = \sqrt{m^2 + p_i^2}
\]

With this new scalar product we can define the physical Hilbert space, and after that go on to find the physical observables. In this case too, for a free particle, every hermitian function of momentum in the kinematical space is hermitian also in the physical space; and again kinematical spacetime coordinates are not sensible physical observables. In the special relativistic contest, however, we have a much broader choice on which one is the right physical coordinate, as opposed to the essential uniqueness of the galilean coordinate.

We could take the same one-parameter family of operators that gave us the galilean physical coordinate:

\[
A^i(T) \equiv x^i - \frac{[H, x^i]}{[H, x^0]} x^0 + \frac{[H, x^i]}{[H, x^0]} T + h.c.
\]

\[
A^0(T) \equiv T, \quad T \in \mathbb{R}
\]

In this combination we can recognize the well know Newton-Wigner position operator, which is the most famous and cited example, speaking of a relativistic position operator. This operator has however know flaws, like not being a covariant vector under a Lorentz boost, and its localized states having exponentially small components that propagates faster than light.

The Newton-Wigner operator, as hinted, is not the only option; the most general family of position operators linear in the kinematical coordinates can be given in the form:

\[
\chi^\mu_\nu(T) = x^\mu - \frac{p^\mu}{p \cdot v} v \cdot x + \frac{p^\mu}{p \cdot v} T + h.c.
\]

This family of operators is labeled, apart from the analogue of the time parameter, by the timelike vector \(v\), which basically specifies the transformation properties of

\footnote{In the relativistic case we have to introduce the inverse of the \(p_0\) operator. This inverse is well defined in the physical space with positive energy - as long as massless particles are not considered - being its spectrum bounded from below.}

\footnote{Otherwise the operator \(p \cdot v\) could not be inverted.}
the \( \chi_v \) coordinate. Posing \( \nu^\mu = \delta_0^\mu \) we obtain again the Newton-Wigner operator, and another notable member of this family is obtained when \( \nu^\mu = p^\mu \). The last one is an important member of the family because it allows us to define a spacetime coordinate which is a vector, as opposed to the Newton-Wigner position operator. In fact while the latter selects an external vector \( \delta_0^\mu \), in the definition of the former we don’t need any vector which is not related to the particle itself.

This covariant coordinate have the form:

\[
\chi^\mu \equiv \chi^\mu_v = x^\mu - \frac{p^\mu}{p^2} p \cdot x + \frac{p^\mu}{p^2} T + h.c. \tag{2.34}
\]

It is useful to note the different commutation rules obeyed by the Newton-Wigner coordinates and the vector coordinates just presented; in the first case we have:

\[
[A^i, A^j] = 0 \tag{2.35}
\]

like in standard Heisenberg quantum mechanics.

In the vector case, on the other hand, we obtain:

\[
[\chi^\mu, \chi^\nu] = \frac{i}{m^2} M^{\mu \nu} = \frac{i}{m^2} (p^\nu x^\mu - p^\mu x^\nu) \tag{2.36}
\]

So we have to trade the commutativity of coordinates with the vector transformation properties of them.

In the last part of the thesis we will use this relativistic theory as a baseline to deform, in order to describe the deformed relativistic symmetries of a non-commutative spacetime. For now we don’t go deeper in the analysis of the special relativistic regime, in that given the covariant nature of the theory there are no conceptual novelties in treating the special relativistic symmetry group with respect to the galilean relativistic one.
Part II

Non-commutative spacetime and its symmetries
Chapter 3

$k$-Minkowski spacetime: Definition

We will see in this chapter how the algebra-coalgebra formalism can be useful in extending the usual notion of commutative spacetime.

The standard, commutative, Minkowski spacetime is simply $\mathbb{R}^{3,1}$: $\mathbb{R}^4$ endowed with the standard metric $\eta = \text{Diag}(-1, 1, 1, 1)$.

As a vector space it can be coordinatized by four real numbers $(x_0, \vec{x})$. Moreover, being the coordinates just real numbers they form a unital associative algebra and a (trivial) Lie algebra, with Lie brackets given by commutators:

$$[x^\mu, x^\nu] = 0$$

We will call the Minkowski Lie algebra $\mathfrak{g}_0$.

The $k$-Minkowski spacetime\[18, 19, 20]\ algebra $\mathfrak{g}_\ell$ can be seen as a deformation of the former Lie algebra by a dimensionful parameter $\ell$:

$$[x^i, x^0] = i\ell x^i \quad [x^j, x^i] = 0 \quad \forall i, j = 1, 2, 3$$

(3.2)

is of course only one of infinitely many possible deformations of the commutative algebra, which can be written as

$$[x^\mu, x^\nu] = i\Upsilon^{\mu\nu}(p, x)$$

with $\Upsilon^{\mu\nu}$ real functions.

Another example of non-commutative spacetime is the so called “Canonical” or $\theta$-Minkowski spacetime\[6, 21]\ it can be obtained from 3.3 with a constant matrix $\theta^{\mu\nu}$. The last spacetime arises both in M-theory in presence of external fields - with $\theta^{\mu\nu}$ the components of a tensor - or in the context of deformed symmetry - with $\theta^{\mu\nu}$ Lorentz scalars.

In this chapter we will turn our attention mostly on $k$-Minkowski spacetime, with its Lie algebra structure. This space has a very nice feature: it has a deep

\[1\text{Historically the deformation was introduced by a parameter } k = \ell^{-1}, \text{ with the dimension of an energy, hence the name “$k$-Minkowski”.}\]

\[2\text{We should speak of a self-adjoint functions, being the non-null commutator incompatible with the standard product in } \mathbb{R} \text{ and so requiring an operator algebra representation for the } x^\mu.\]
connection with the known symmetry group $\kappa$-Poincaré. As we will see one way to build $\kappa$-Minkowski is to look for the so-called homogeneous space of $\kappa$-Poincaré; it admits a set of translation parameters with the same defining commutation relations of coordinates, so the parameters space is a copy of the spacetime, like in the commutative case. We will see that our search for symmetries and conserved charge will drive us a little away from this road, but it is still worth noting this peculiarity of $\kappa$-Minkowski. In any case a vast literature can be found in any of the cited models of non-commutative spacetime, also the canonical one.

3.1 Functions on the $\kappa$-Minkowski algebra and unitary group

To define a field theory on this spacetime we have as a first step to define a field: in Minkowski spacetime we have functions (distributions, in the most general case) from $\mathbb{R}^4$ to $\mathbb{C}$. In the free field expansion we restrict to functions that can be written as a continuous superposition of plane waves:

$$\psi(x) = \int d^4k \tilde{\psi}(k)e^{ikx}$$

(3.4)

So we can take the plane waves as a basis for our space of functions.

We try to extend the definition of field in $\kappa$-Minkowski by the extension of 'plane wave' functions to the $\kappa$-Minkowski algebra: we seek a correspondence

$$e^{ikx} \rightarrow e^{ik\hat{x}}$$

(3.5)

between complex exponentials of real coordinates and complex exponential operators generated by the $\kappa$-Minkowski algebra. These correspondences (the choice is not unique due to the non commutativity of the $\hat{x}$'s) are called Weyl maps, and their image is the unitary group generated by the $\kappa$-Minkowski algebra.

To define the exponential map in our algebra, however, we must face convergence problems in the defining series of the exponentials: we don’t know how to take the limit $N \rightarrow \infty$ in the series

$$e^{ik\mu\hat{x}^\mu} = \lim_{N \rightarrow \infty} \sum_{n=0}^{N} \frac{(ik\mu\hat{x}^\mu)^n}{n!}$$

(3.6)

What we can do without problems is to define the polynomial algebra over $\kappa$-Minkowski, that being an associative algebra in addition to a Lie’s one permits us to take arbitrary powers and linear combinations of its elements.

We call this algebra $\mathcal{P}_\ell$, and it contains every element of the form

$$P(x) = \sum_{n \in \mathbb{Z}^4} k_{n\ell} p_{n\ell}(x_0, \vec{x})$$

(3.7)

with $k_{n\ell}$ a compact-supported function from $\mathbb{Z}^4$ to $\mathbb{C}$ and $p_{n\ell}(x_0, \vec{x})$ a generic monomial of degree $n_0$ in $\hat{x}_0$, $n_1$ in $\hat{x}_1$, ecc...
For a given \( \vec{n} \) we have
\[
\lim_{\ell \to 0} p_\vec{n}(x_0, \vec{x}) = x_0^{n_0} x_1^{n_1} x_2^{n_2} x_3^{n_3}
\]
but, for example, in the non commutative case we have both
\[
x_0^{n_0-1} x_1^{n_1} x_2^{n_2} x_3^{n_3} \quad \text{and} \quad x_1^{n_1} x_2^{n_2} x_3^{n_3} x_0^{n_0},
\]
that are different due to the nontrivial commutation rules between coordinates.

We can define Weyl maps on \( \wp_\ell \) by simply giving a choice of the ordering in the elements of all the \( p_\vec{n}(x_0, \vec{x}) \): for example we can define the time-to-the-right map
\[
\Omega_R : \wp \to \wp \ell
\]
\[
\begin{align*}
x_0^{n_0} x_1^{n_1} x_2^{n_2} x_3^{n_3} & \to \hat{x}_1 \hat{x}_2 \hat{x}_3 \hat{x}_0
\end{align*}
\]
(with \( \wp \) the space of polynomials in \( \mathbb{R}^4 \)).

To define the generic non-commutative function we must upgrade our equipment from simple polynomials to at least plane waves, that are infinite series. Instead of studying the limit process we switch from the abstract algebra setting for the \( \hat{x}_\mu \) to their representations as hermitian operators on an Hilbert space. In this setting we can use Stone’s theorem to assert that the plane waves \( e^{i k_\mu \hat{x}_\mu} \) are strongly continuous unitary operators on the same space on which the \( \hat{x}_\mu \) are defined. Then we use the existence of a functional calculus to define a Fourier transform:
\[
f(\hat{x}) = \int d^4 x f(x) E(x)
\]
where \( x_\mu \) are real numbers in the spectrum of the hermitian \( \hat{x}_\mu \), \( f(x) \) is a standard integrable function of real numbers, and \( E(x) \) is the orthogonal projector on the eigenspaces with eigenvalue \( x_\mu \) (with the right order).

So, given a function
\[
f(x) = \int d^4 k \tilde{f}(k) e^{i k x}
\]
we can define its non-commutative counterpart by Fourier trasform as
\[
f(\hat{x}) = \int d^4 x \left( \int d^4 k \tilde{f}(k) e^{i k x} \right) E(x) = \int d^4 k \tilde{f}(k) e^{i \hat{k} \hat{x}}
\]
(3.11)
Where we changed the order of integration thanks to the finiteness of the integrals, given by the fact that \( f \) is Fourier-trasformable.

The definition of a given non-commutative function must not depend on the choice of a Weyl maps, so if we change the type of map we have to adapt the choice of the counter-image of the function, in order to leave the image unchanged:
\[
f(\hat{x}^\mu) = \int d^4 k \tilde{f}_\alpha(k_\mu) \Omega_\alpha(e^{i k_\mu x})(x_\mu) = \Omega_\alpha(f_\alpha(x))
\]
where \( \Omega_\alpha \) denote the generic Weyl map.

---

3 In this setting all the order ambiguities that we have seen for the polynomial space are dumped on the orthogonal projectors: for example for \( [\hat{x}_1, \hat{x}_0] \) we have
\[
[\hat{x}_1 \hat{x}_0] = \int dx_1 dx_0 x_1 x_0 [E(x_1), E(x_0)].
\]
We focus now on the multiple possible choices for a Weyl map: the only property we ask for it is that the map is linear - so we can extend a map from plane waves to generic functions - and that it reduces to the classical plane wave in the limit $\ell \to 0$.

It’s easy to see that we are left with complete liberty about the position of the time coordinate in the exponentials: in $\kappa$-Minkowski

$$e^{-ik_0 \hat{x}_0} e^{ik \hat{x}} \neq e^{ik \hat{x}} e^{-ik_0 \hat{x}_0} \quad (3.14)$$

because of the commutation relations between coordinates, but both of the above expressions have the same classical limit.

We have infinite possible choices on the position of the time coordinate, and so infinitely many Weyl maps, one for each choice. The most used ones are:

$$\Omega_S(e^{ikx}) = e^{ik \hat{x}^\mu \hat{x}_\mu} \quad (3.15)$$

$$\Omega_R(e^{ikx}) = e^{ik \hat{x} \cdot \hat{x}} e^{-ik_0 \hat{x}_0} \quad (3.16)$$

$$\Omega_L(e^{ikx}) = e^{-ik_0 \hat{x}_0} e^{ik \hat{x} \cdot \hat{x}} \quad (3.17)$$

They are called, respectively, Weyl-ordered, time-to-the-right, and time-to-the-left map. Obviously all physical results should be independent of the choice of the map.

We return for a moment on polynomials to prove a remarkable feature of Weyl maps. The feature we want to prove is their invertibility, despite the space of non-commutative polynomials is much bigger than the commutative one. The reason is that a Weyl maps consists in fixing a product order; this order-fixing induces an equivalence relation in the space of non-commutative polynomials, and the quotient space is equivalent to that of commutative polynomials. To prove the sentences given above we prove injectivity and surjectivity of every Weyl map: the injectivity is straightforward, because to a single commutative polynomial we can associate only non-commutative polynomials in the same equivalence class given by (3.2), and an equivalence class can be the image of only a single commutative polynomial. For surjectivity we note the existence of a relation between two different polynomials in the non-commutative case, given by the commutation rules (3.2):

$$p_{\vec{n}} = \alpha q_{\vec{n}} + \sum_{m_0<n_0} r_{\vec{m}}(x) \quad (3.18)$$

with $p_{\vec{n}}$ a generic non-commutative polynomial, and $q_{\vec{n}}(x) = \Omega(x^{m_0}_{0} x^{n_1}_{1} x^{n_2}_{2} x^{n_3}_{3})$; we can apply recursively the relation (3.18), so we obtain the relation

$$p_{\vec{n}}(x) = \alpha \Omega(x^{m_0}_{0} x^{n_1}_{1} x^{n_2}_{2} x^{n_3}_{3}) + \sum_{m_0<n_0} r_{\vec{m}}(x) \quad (3.19)$$

applying again the same relation, with $p_{\vec{n}} = \sum_{m_0<n_0} r_{\vec{m}}(x)$, we can further reduce the degree of the polynomial, and recursively we obtain:

$$p_{\vec{n}}(x) = \sum_{m_0\leq n_0} \alpha_{\vec{m}} \Omega(x^{m_0}_{0} x^{m_1}_{1} x^{m_2}_{2} x^{m_3}_{3}) \quad (3.20)$$

and by linearity:
3.1 Functions on the $\kappa$-Minkowski algebra and unitary group

$$p_{\vec{n}}(x) = \Omega \left( \sum_{m_0 \leq n_0} \alpha_{\vec{m}} x_0^{m_0} x_1^{m_1} x_2^{m_2} x_3^{m_3} \right)$$

(3.21)

so we have constructed from a non-commutative polynomial the commutative counterimage through a generic Weyl map, proving the surjectivity of such a map.

In the previous definition of Fourier transform we relied on a functional calculus; if we limit ourselves in the space of polynomials we can also define it without convergence problems: a parameter-dependent polynomial can be written as

$$P(\hat{x}, \alpha) = \sum_{\vec{n}} f_{\vec{n}}(\alpha) p_{\vec{n}}(\hat{x})$$

(3.22)

so, its integral with respect to the parameter is:

$$\int d\alpha P(\hat{x}, \alpha) = \int d\alpha \sum_{\vec{n}} f_{\vec{n}}(\alpha) p_{\vec{n}}(\hat{x})$$

(3.23)

that (with simple conditions on $f_{\vec{n}}$) can be rewritten as

$$\int d\alpha P(\hat{x}, \alpha) = \sum_{\vec{n}} \left( \int d\alpha f_{\vec{n}}(\alpha) \right) p_{\vec{n}}(\hat{x}) = \sum_{\vec{n}} g_{\vec{n}} p_{\vec{n}}(\hat{x}) = G(\hat{x})$$

(3.24)

We rely so strongly on plane waves because we can define on them various properties we want to hold for generic functions, and then extend those properties by linearity; for example, being the algebra of coordinates non-commutative we don’t have a standard, commutative algebra structure in our function space, but a deformation of the standard product:

$$f(\hat{x}^\mu) g(\hat{x}^\nu) \neq \Omega(f(x^\mu) g(x^\nu))$$

(3.25)

We have to introduce the so-called Moyal product, a deformation of the standard pointwise product, to express the non commutative product of $\kappa$-Minkowski function in terms of commutative coordinates:

$$f(\hat{x}^\mu) g(\hat{x}^\nu) = \Omega(f(x^\mu) \star g(x^\nu))$$

(3.26)

or, equivalently\[4\]

$$f(x^\mu) \star g(x^\nu) = \Omega^{-1}(f(\hat{x}^\mu) g(\hat{x}^\nu))$$

(3.27)

To handle calculations with such a product, as we said, we must rely on its definition for plane waves, and then extend to a generic function by linearity; for example for two exponentials in time-to-the-right map we have\[5\]

\[4\]Note that the inverse of the Weyl maps is well defined, having proved its bijectivity.

\[5\]We want to use the Baker-Campbell-Hausdorff formula, for which we have to limit ourselves on analyticity subdomains for our operators.
where we have introduced the deformed momentum composition rule enforced by the non-commutativity of coordinates:

\[(k \oplus q)_0 = k_0 + q_0\] (3.31)
\[(k \oplus q)_i = k_i + e^{-\ell k_0}q_i\] (3.32)

or, more compactly:

\[(k \oplus q)_\mu = k_\mu + e^{-\ell k_0}(1 - \delta_\mu^0)q_\mu\] (3.33)

### 3.1.1 Integrals in κ-Minkowski

Another concept we want to introduce is that of integration over spacetime (or, eventually, its subspaces); we want a linear map that sends κ-Minkowski functions in complex numbers, and that reduces to the standard Riemann integral in the \(\ell \to 0\) limit.

To find such a map we can again exploit the plane waves, and the Fourier transform; assuming that spacetime integrals are also well-behaved under interchange of integration order, we can write:

\[
\int d^4\hat{x} f(\hat{x}^\mu) = \int d^4k \hat{f}(k_\mu) \int d^4\hat{x} \Omega(e^{ik_\mu x^\mu})
\] (3.34)

So, again, the only definition we need is that of integral of plane waves; a condition this definition has to satisfy is:

\[
\lim_{\ell \to 0} \int d^4\hat{x} \Omega_\alpha(e^{ik_\mu x^\mu}) = \delta^4(k)
\] (3.35)

This tells us that these integrals have to be functions of \(\ell\) with values distributions in \(k\), that reduce to the dirac delta in the \(\ell = 0\) limit, and that the Weyl map dependece of the distribution must compensate with that of the Fourier components to give a map-independent integral.

In general the integral of a plane wave must depend on the Weyl map of our choice, so we can’t simply assume that also the integral of a non-commutative plane wave is a dirac delta. It comes in our help, however, another nice feature of the Weyl maps: the zero-momentum Fourier component of a function is independent from the choice of Weyl map (we couldn’t say that apriori, given that the Fourier components has to be changed to ensure the invariance of the image of the Weyl map).

To prove the above statement we have to express the relationship between Fourier components linked to different Weyl maps, and to do that we have in turn to express the relation between various Weyl maps; for example, in passing from time-to-the-right to time-to-the-left map, we have:
3.1 Functions on the $\kappa$-Minkowski algebra and unitary group

$$\Omega_r(e^{ik_\mu x^\mu}) = e^{ikx} e^{-ik_0 x_0}$$ (3.36)

$$= e^{-ik_0 x_0} e^{i\ell k \cdot \vec{x}}$$ (3.37)

$$= \Omega_l(e^{ik'_\mu x'\mu})$$ (3.38)

with $k'_\mu = e^{\ell k_0 (1-\delta_0^\mu)} k_\mu$.

So, in the Fourier transform, we have:

$$\Omega_r(f_r(x)) = \int d^4k \tilde{f}_r(k) \Omega_r(e^{ik_\mu x^\mu})$$ (3.39)

$$= \int d^4k \tilde{f}_r(k) \Omega_l(e^{ik'_\mu x'\mu})$$ (3.40)

$$= \int d^4k' e^{-3\ell k_0} \tilde{f}_r(e^{-\ell k_0 (1-\delta_0^\mu)} k') \Omega_l(e^{ik'_\mu x'\mu})$$ (3.41)

$$= \int d^4k' e^{-3\ell k_0} \tilde{f}_r(e^{-\ell k_0 (1-\delta_0^\mu)} k') \Omega_l(e^{ik'_\mu x'\mu})$$ (3.42)

$$= \int d^4k' \tilde{f}_l(k') \Omega_l(e^{ik'_\mu x'\mu})$$ (3.43)

$$= \Omega_l(f_l(x))$$ (3.44)

So we derive, for the Fourier components:

$$\tilde{f}_l(k) = e^{-3\ell k_0} \tilde{f}_r(e^{-\ell k_0 (1-\delta_0^\mu)} k)$$ (3.45)

This result, that we have seen as an example, is more general: the effect of changing Weyl map is to multiply the Fourier components of a function and its arguments by exponentials of the time component of the momentum, so the zero momentum component is always unchanged.

Given that the zero momentum component of every function is independent of the Weyl map we pick to represent that function, there is nothing preventing us from picking up a trivial deformation for the delta function; so we can define the integral of a plane wave as:

$$\int d^4x \Omega_\alpha(e^{ik_\mu x^\mu}) = \delta^3(k)$$ (3.46)

and, consequently, define the spacetime integral of a function as

$$\int d^4x f(x) = \tilde{f}(0)$$ (3.47)

Similarly, we want to define also a three dimensional integral - for example it’s necessary for the definition of a conserved charge associated to a current - in the most general way possible. We have that also the 3-dimensional integral of plane waves has to be a function of $\ell$ with values distribution in $k$:

$$\int d^3\hat{x} \Omega_\alpha(e^{ik_\mu x^\mu}) = \gamma_\alpha(\vec{k}, k_0, \vec{x}_0, \ell) \xrightarrow{\ell \rightarrow 0} \delta^3(\vec{k}) e^{-ik_0 x_0}$$ (3.48)
In this case, however, we can’t give a universal definition for the integral, that is independent from the Weyl map chosen; we can easily see this fact with the example of the time-to-the-right and time-to-the-left maps:

\[ \Omega_r(e^{ik_\mu x^\mu}) = \Omega_l(e^{i\ell k_0(1-\delta^0_0)k_\mu x^\mu}) \]  (3.49)

so, for the three dimensional integral to be independent of the choice of Weyl map we must have

\[ \gamma_r(\vec{k}, k_0, \hat{x}_0, \ell) = \gamma_l(e^{i\ell k_0} \vec{k}, k_0, \hat{x}_0, \ell) \]  (3.50)

If we choose this distribution to be the dirac delta in the time-to-the-right basis,

\[ \gamma_r(\vec{k}, k_0, \hat{x}_0, \ell) = \delta^3(\vec{k})e^{-ik_0x_0} \]  (3.51)

it can be the same in the time-to-the-left one:

\[ \gamma_l(\vec{k}, k_0, \ell) = \gamma_r(e^{-i\ell k_0} \vec{k}, k_0, \ell) = \delta^3(e^{-i\ell k_0} \vec{k})e^{-ik_0x_0} = e^{i\ell k_0} \delta^3(\vec{k})e^{-i\ell k_0x_0} \]  (3.52)

In the definition of space integrals there is an intrinsic ambiguity linked to the choice of the basis of exponentials we pick. In this thesis work, however, the only spatial integrals we have are those of conserved currents, that are (as we will prove) time independent, so subject to the condition \( e^{P_0}Q = Q \).

### 3.2 Symmetries of \( \kappa \)-Minkowski spacetime

To perform a Noether analysis in our spacetime we have to find the associated symmetry group; we already know what that group is in the limit \( \ell \to 0 \): the symmetry group of Minkowski space, i.e. the Poincaré group.

The introduction of the invariant length scale \( \ell \) seems to break the Lorentz covariance of the theory: after a boost the commutation rules between coordinates should change in order to respect length contraction. It’s similar to the classical case, in which the constant speed of light seems to break the galilean group of symmetry, and in the same way the symmetries can be deformed to be compatible with the new invariant quantity.

The first deformation of the Poincaré algebra was obtained as a limit of the so-called q-deformation of the anti de-sitter algebra \( U(so(3,2)) \): the q-deformation introduces a dimensionless parameter \( q \), and then we can take the limit of the de-Sitter radius \( R \) going to infinity keeping a dimensional quantity \( \ell \) constant:

\[ \lim_{R \to \infty, q \to 1} R \log(q) = \kappa^{-1} = \ell \]  (3.55)

In this way we can introduce an invariant length as a deformation of the Poincaré algebra, that reflects the intuition of a minimal physical length shared by many

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\( ^6\ell \) is a Lorentz scalar, and it’s the same in every reference frame.
quantum-gravity models; the resulting deformed algebra is the so called $\kappa$-Poincaré algebra.

Now we see more precisely why and how to derive the $\kappa$-Poincaré algebra directly from the $\kappa$-Minkowski spacetime; the latter is not invariant under a classic lorentz trasformation, for in $\kappa$-Minkowski spacetime we have for example:

$$\hat{x}^1 \hat{x}^0 = (\hat{x}^0 + i\ell)\hat{x}^1$$

(3.56)

the r.h.s. of the equation contains the sum of a second order and a first order tensor component with respect to the Lorentz group, so it cannot be a covariant equation. From this simple observation we can deduce that the Poincaré algebra is not the right one to describe the symmetries of our spacetime.

### 3.2.1 Translations sector

To describe the symmetries of $\kappa$-Minkowski we start from the condition that, like in the classical case, our spacetime and the translation sector of its symmetry group have to be dual Hopf algebras. In the commutative case we have also that the action on the unitary group is simply to isolate the Fourier parameter associated to the plane wave:

$$P_\mu e^{ik_\rho x^\rho} = -i\partial_\mu e^{ik_\rho x^\rho} = k_\mu e^{ik_\rho x^\rho}$$

(3.57)

Imposing this condition to hold also in the non-commutative case we see immediately that the translation generators depend on the Weyl map we pick for the exponentials (basis of the space of functions on $\kappa$-Minkowski):

$$P^\alpha_\mu \Omega_\alpha (e^{ik_\rho x^\rho}) = k_\mu \Omega_\alpha (e^{ik_\rho x^\rho})$$

(3.58)

so we have, for example:

$$P^\mu_\mu \Omega_\mu (e^{ik_\rho x^\rho}) \neq P^\rho_\rho \Omega_\rho (e^{ik_\rho x^\rho}) = k_\mu \Omega (e^{ik_\rho x^\rho})$$

(3.59)

To specify the action of translation generators on exponentials is enough to derive their action on a generic function, thanks to linearity of the Fourier trasform:

$$P^\mu_\mu f(\hat{x}) = \int d^4k \hat{f}_\alpha (k) P^\alpha_\mu \Omega_\alpha (e^{ik_\rho x^\rho})$$

(3.60)

$$= \int d^4k \hat{f}_\alpha (k) k_\mu \Omega_\alpha (e^{ik_\rho x^\rho})$$

(3.61)

With this definition we see that the coproduct rule for the $P_\mu$ must be a non trivial one; the group structure of the unitary group must be covariant with respect to the action of a translation generator, so we have (for example in the time-to-the-right basis of exponentials):

---

7Historically the $\kappa$-Minkowski spacetime was derived as homogeneous space associated to the $\kappa$-Poincaré algebra, that is, the dual Hopf algebra to the translation sector of $\kappa$-Poincaré.

8We are imposing the property of $\ell$ of being an invariant length.

9We say the translation generators have classical action on exponentials.
that means that for the translation sector to be the dual algebra of the unitary

group (and consequently to the \( \kappa \)-Minkowski algebra that generates it) it has to

have coproduct rule:

\[
\Delta(P_\mu^r) = P_\mu^r \otimes \mathbb{1} + e^{-\ell P_0^r (1-\delta_0^\mu)} \otimes P_\mu^r
\]  

(3.64)

This definition of translation generators, as we said, is dependent on the basis of

exponential we choose; for example in the time-to-the-left basis we would have:

\[
e^{-i k_0 \hat{x}_0} e^{i \hat{k} \cdot \hat{x}} e^{-i q_0 \hat{x}_0} e^{i \hat{q} \cdot \hat{x}} = e^{-i (k+q) \hat{x}_0} e^{i (\ell q_0 \hat{k} + \hat{q}) \cdot \hat{x}}
\]  

(3.65)

and consequently the following coproduct rule for the momenta would have to hold:

\[
\Delta(P_\mu^l) = P_\mu^l \otimes e^{\ell P_0^l (1-\delta_0^\mu)} + \mathbb{1} \otimes P_\mu^l
\]  

(3.66)

We can merge the last two basis in a more comprehensive, one-parameter family

of Weyl maps given by:

\[
\Omega_\varepsilon(e^{ik_\mu x_\mu}) = e^{-i\varepsilon k_0 \hat{x}_0} e^{i \hat{k} \cdot \hat{x}} e^{-i(1-\varepsilon) k_0 \hat{x}_0}
\]  

(3.67)

that gives for the translation generators the one-parameter family:

\[
\Delta(P_\mu^\varepsilon) = P_\mu^\varepsilon \otimes e^{\ell \varepsilon P_0^\varepsilon (1-\delta_0^\mu)} + \mathbb{1} \otimes P_\mu^\varepsilon
\]  

(3.68)

In all these Weyl maps the time component of the momentum has the same

coproduct rule, and it is in fact the same operator; for it we have:

\[
P_0^{\alpha} \Omega_\beta(e^{ik_\mu x_\mu}) = k_0 \Omega_\beta(e^{ik_\mu x_\mu}) \quad \forall \alpha, \beta
\]  

(3.69)

where \( \alpha \) and \( \beta \) indicates two different basis of exponentials equivalent modulo

commutation relations \( [3.2] \).

The deep reason for this fact is in the behaviour of the time coordinate in the

\( \kappa \)-Minkowski defining commutation relations; they are:

\[
[\hat{x}_i, \hat{x}_0] = i\ell \hat{x}_i
\]  

that in the adjoint representation formalism can be seen as:

\[
ad_{\ell \hat{x}_0}(\hat{x}_i) = i\ell \hat{x}_i
\]  

(3.70)

So we see that the application of the commutation relations \( [3.2] \) amounts only
to the application of a linear operator to the \( \hat{x}_i \); for example the passage from
time-to-the-right to time-to-the-left basis can be seen ad\( \ell \).

\footnote{All this results are derived explicitly in the appendix.}
Given the basis independence of $P_0$ we can write it without indices to remark what Weyl map we are using; the spatial momentum components will maintain this ambiguity, but we will see in the next part that it doesn’t affect physical translations, made up by momentum components and translation parameters. Assuming for now as proved the fact that this ambiguity has no physical consequences, we can choose once and for all a basis for the exponentials, and keep it for the rest of the thesis. We will choose the time-to-the-right basis, as it simplifies some calculation, and from now on we will omit the $(r)$ index in $P^r_\mu$; whenever we write $P_\mu$ we are assuming it is the time-to-the-right momentum, with coproduct:

$$\Delta(P_\mu) = P_\mu \otimes 1 + e^{-\ell P_0(1-\delta^0_\mu)} \otimes P_\mu$$

### 3.2.2 Rotations sector

Once we have found the translation sector as the dual Hopf algebra to $\kappa$-Minkowski, we can find the Lorentz sector of the $\kappa$-Poincaré group asking for it to form a close algebra and coalgebra with translations generators, and that its limit for $\ell \to 0$ is the Lorentz sector of ordinary Poincaré group. It’s apparent from the relations [3.2] that the $\kappa$-Minkowski algebra is covariant with respect to spatial rotations also in their classical form: $\hat{x}_0$ and $\ell$ are both scalars with respect to spatial rotations, so we are not yet forced to deform this sector of the algebra to adapt it to the non-commutative spacetime. We will try to keep the classical definition for rotations generators, and see how they get along with Weyl maps and translations generators; for commutative rotations generators we have:

$$R_i = \epsilon_{ijk} x_j P_k$$

To ask for a classical action it means that their action has to commute with the Weyl maps, that is, for example in time-to-the-right basis:

$$R^r_i f(\hat{x}) = R^r_i \Omega_r(f_r(x)) = \Omega_r(\epsilon_{ijk} x_j (-i\partial_k) f_r(x))$$

In the last member we can pull out the spatial coordinates that are to the left, and write the derivative inside the Weyl map as the momentum acting outside:

$$\Omega_r(\epsilon_{ijk} x_j (-i\partial_k) f_r(x)) = \epsilon_{ijk} \hat{x}_j P^r_k f(\hat{x})$$

---

11 $\epsilon_{ijk}$ is the usual Levi-Civita symbol, with $\epsilon_{123} = 1$. 

---

$$e^{ik_i \hat{x}_i} e^{-ik_0 \hat{x}^0} = e^{-ik_0 \hat{x}^0} e^{if(k_0 \hat{x}^0, k_i \hat{x}_i)}$$

$$\Rightarrow e^{if(k_0 \hat{x}^0, k_i \hat{x}_i)} = e^{ik_0 \hat{x}^0} e^{ik_i \hat{x}_i} e^{-ik_0 \hat{x}^0}$$

$$= e^{ad_{ik_0 \hat{x}^0}} e^{ik_i \hat{x}_i}$$

$$= e^{ie^{-ik_0 \hat{x}^0} k_i \hat{x}_i}$$

$$= e^{ie^{-ik_0 \hat{x}^0} k_i \hat{x}_i}$$

$$\Delta(P_\mu) = P_\mu \otimes 1 + e^{-\ell P_0(1-\delta^0_\mu)} \otimes P_\mu$$

$$\Omega_r(\epsilon_{ijk} x_j (-i\partial_k) f_r(x)) = \epsilon_{ijk} \hat{x}_j P^r_k f(\hat{x})$$

---

$\epsilon_{ijk}$ is the usual Levi-Civita symbol, with $\epsilon_{123} = 1$. 

---
It seems we introduced the same ambiguities we had in the translation case, writing the rotation generators as functions of translation ones; we will demonstrate that as a matter of fact rotation generators are independent of the Weyl map in use, at least for maps of the one-parameter family introduced in the previous subsection.

In time-to-the-left map\textsuperscript{12} we have:

\[ R^i_1 \Omega_l(f_i(x)) = \Omega_l(\epsilon_{ijk} x_j(-i \partial_k) f_i(x)) = \epsilon_{ijk} \left[ P^i_k f(\hat{x}) \right] \hat{x}_j \quad (3.80) \]

and in the \( \epsilon \) maps:

\[ R^i_\epsilon \Omega_\epsilon(f_\epsilon(x)) = \Omega_\epsilon(\epsilon_{ijk} x_j(-i \partial_k) f_\epsilon(x)) \quad (3.81) \]

in this case, however, we can’t strip out no spatial coordinates unless \( \epsilon = 0 \) or 1.

We now show the independence of \( R^i_\epsilon \) from the \( \epsilon \) index:

\[
R^i_\epsilon f(\hat{x}) = \Omega_\epsilon(\epsilon_{ijk} x_j(-i \partial_k) f_\epsilon(x)) = \\
= \epsilon_{ijk} \int d^4q k \hat{f}_\epsilon(q) \Omega_\epsilon(x_j e^{iq_\mu x^\mu}) = \\
= \epsilon_{ijk} \int d^4q k \hat{f}_\epsilon(q) e^{-i\epsilon q_\mu x^\mu} x_j e^{iq_\mu x^\mu} e^{-i(\epsilon-\delta_0^\mu) q_\mu x^\mu} = \\
= \epsilon_{ijk} x_j \int d^4q k e^{-i\epsilon q_\mu x^\mu} \hat{f}_\epsilon(q) e^{i(\epsilon-\delta_0^\mu) q_\mu x^\mu} = \\
= \epsilon_{ijk} x_j \int d^4q k e^{i\epsilon q_\mu x^\mu} \hat{f}_\epsilon(q) e^{-i(1-\epsilon) q_\mu x^\mu} = \\
= R^i_\epsilon f(\hat{x}) \quad (3.82)
\]

where we used the Fourier components relation:

\[ e^{i\epsilon q_\mu x^\mu} \hat{f}_\epsilon(q) e^{-i(1-\epsilon) q_\mu x^\mu} = \tilde{f}_\epsilon(q) \quad (3.83) \]

and the change of variables

\[ q_0' = q_0, \quad q_i' = e^{-\epsilon q_0} q_i \quad (3.84) \]

Having proved that every rotation generator in the \( \epsilon \) family is the same as the time-to-the-right one we have established that indeed they are independent of \( \epsilon \). In particular we have\textsuperscript{13}

\[ R^i_\epsilon f(\hat{x}) = R^i_\epsilon f(\hat{x}) \quad (3.85) \]

Using the basis independence of \( R^i_\epsilon \) we can omit the “Weyl index” on it, and show that it has trivial coproduct, like the time component of the momentum:

\textsuperscript{12}For which we are free to move spatial coordinates outside the map to the right.

\textsuperscript{13}It would have been possible to see this in a simpler way:

\[ R^i_\epsilon f(\hat{x}) = \epsilon_{ijk} \left[ P^i_k f(\hat{x}) \right] \hat{x}_j = \epsilon_{ijk} \hat{x}_j e^{i\epsilon q_0} \left[ P^i_k f(\hat{x}) \right] = \epsilon_{ijk} \hat{x}_j \left[ P^i_k f(\hat{x}) \right] = R^i_\epsilon f(\hat{x}) \]
\[ R_i \left[ f(\hat{x})g(\hat{x}) \right] = R_i^r \left[ f(\hat{x})g(\hat{x}) \right] = \epsilon_{ijk}\hat{x}^j P_k^r \left[ f(\hat{x})g(\hat{x}) \right] \] (3.86)

and it’s easy to show that the operator \( \hat{x}^j P_k^r \) obeys Leibniz rule:

\[ \hat{x}^j P_k^r \left[ f(\hat{x})g(\hat{x}) \right] = \left[ \hat{x}^j P_k^r f(\hat{x}) \right] g(\hat{x}) + \hat{x}^j \left[ e^{-\epsilon P_0} f(\hat{x}) \right] P_k^r g(\hat{x}) \] (3.87)

then we use the commutation rule between coordinates and exponentials\(^{14}\) \( x_i e^{-ik_0 \hat{x}^0} = e^{-ik_0 \hat{x}^0} e^{ik_0 x_i} \) to conclude:

\[ \hat{x}^j P_k^r \left[ f(\hat{x})g(\hat{x}) \right] = \hat{x}^j P_k^r f(\hat{x}) g(\hat{x}) + f(\hat{x}) \left[ \hat{x}^j P_k^r g(\hat{x}) \right] \] (3.88)

That implies, for the coproduct formula:

\[ \Delta(R_i) = R_i \otimes 1 + 1 \otimes R_i \] (3.89)

The last useful algebraic information about \( R_i \) is about their commutator with translation generators and between themselves; it’s easy to show that both of them remain classical:

\[ [R_i, P_0] = 0 \] (3.90)

\[ [R_i, P_0^r] = \epsilon_{ijk} \left[ \hat{x}^j P_k^r P_l^i - P_l^i \hat{x}^j P_k^r \right] \] (3.91)

\[ = \epsilon_{ijk} \left[ \hat{x}^j P_k^r P_l^i - \left( P_l^i \hat{x}^j \right) e^{\epsilon P_0} P_k^r - \hat{x}^j P_k^r P_l^i \right] \] (3.92)

\[ = -\epsilon_{ijk} \left( P_l^i \hat{x}^j \right) e^{\epsilon P_0} P_k^r \] (3.93)

\[ = i\epsilon_{ikl} P_k^r \] (3.94)

where we used the obvious fact that translation generators commute, and the relation between \( \varepsilon \) basis momenta:

\[ P_i^r = e^{\ell(\varepsilon - \varepsilon')} P_0^r \] (3.95)

(remember that \( P_i^{r=0} = P_i^r \)) So we see that momenta retain their vector nature with respect to rotations.

Finally, we have to compute the commutator between different rotation generators (here we can pick whatever basis we like for translations, given the unicity of rotations):

\[ [R_i, R_m] = \epsilon_{ijk}\epsilon_{nmn} \left[ \hat{x}^j P_k, \hat{x}^n P_l \right] \]

\[ = \epsilon_{ijk}\epsilon_{nmn} \left[ \hat{x}^j \left[ P_k, \hat{x}^n \right] P_l + \hat{x}^n \left[ \hat{x}^j, P_l \right] P_k \right] \]

\[ = \epsilon_{ijk}\epsilon_{nmn} \left[ \hat{x}^j (-i) \delta^k_n P_l + \hat{x}^n (i) \delta^j_l P_k \right] \]

\[ = i\epsilon_{ilm} R_l \] (3.96)

That reassure us on the fact that also \( R_i \) retain vector nature, and that they form a closed algebra, both between themselfer and along with translation generators.

\(^{14}\)See the appendix.
3.2.3 Boosts sector

The hardest part of our work (in fact, the only non trivial part) is to find the suitable deformation of the boosts sector of the Poincaré algebra. The form of commutation relations (3.2) is clearly noncovariant under classical boosts, so we cannot retain their action.

To find the suitable deformation we impose two conditions on boosts generators; the first will be the right covariance properties of the generators under rotations: we will ask them to form a tridimensional vector $N_i$, and thus that their commutators with spatial and time components of the momentum form respectively a two index tensor and a vector. The second condition will be the closure of the algebra formed only by rotations and boosts, as in the classical case.

The most general commutation relations between boosts and translations generators in agreement with the first condition are:

$$\left[N_i, P_j\right] = \left[iA(\ell P_0, \ell^2|\vec{P}|^2)\ell P_i + iB(\ell P_0, \ell^2|\vec{P}|^2)\delta_{ij}\ell^{-1} + iC(\ell P_0, \ell^2|\vec{P}|^2)\epsilon_{ijk}P_k\right]$$ \hspace{1cm} (3.97)

$$\left[N_i, P_0\right] = iD(\ell P_0, \ell^2|\vec{P}|^2)P_i$$ \hspace{1cm} (3.98)

where the four functions $A$, $B$, $C$ and $D$ are dimensionless scalars (and thus functions of dimensionless scalars only); enforcing the closure of the Lorentz sector as an algebra and of the whole $\kappa$-Poincaré group as a coalgebra we obtain the so-called bycrossproduct basis, first derived by Majid and Ruegg [22]

$$\left[N_i, P_0\right] = iP_i$$ \hspace{1cm} (3.99)

$$\left[N_i, P_j\right] = i\delta_{ij} \left(\frac{1 - e^{-2\ell P_0}}{2\ell} + \frac{\ell}{2}|\vec{P}|^2\right) - i\ell P_i P_j$$ \hspace{1cm} (3.100)

Now we will look for an expression for boosts as similar as possible to the classical one; in particular, for covariance under rotations:

$$N_i = -\hat{x}_0 A(P_0, P_i) + \hat{x}_i B(P_0, P_i)$$ \hspace{1cm} (3.101)

where $A$ and $B$ are generic functions of momenta, compatible with covariance, dimensional analysis, and classical limit. With a simple substitution in the commutator we have:

$$\left[N_i, P_0\right] = -[\hat{x}_0, P_0] A(P_0, P_i) = iA(P_0, P_i)$$ \hspace{1cm} (3.102)

$$\left[N_i, P_j\right] = -[\hat{x}_0, P_j] A(P_0, P_i) + [\hat{x}_i, P_j] B(P_0, P_i)$$ \hspace{1cm} (3.103)

where we used the “Heisenberg relations”: 

3.2 Symmetries of $\kappa$-Minkowski spacetime

\[ [P_0, \hat{x}_0] = i \] 
\[ (3.104) \]
\[ [P_i, \hat{x}_j] = -i \delta_{ij} \] 
\[ (3.105) \]
\[ [P_i, \hat{x}_0] = -\ell P_i \] 
\[ (3.106) \]

that in turn can be derived from coproducts.

Comparing (3.103) with (3.100) we finally obtain:

\[ N_i = -\hat{x}_0 P_i + \hat{x}_i \left( 1 - e^{-2\ell P_0} + \frac{\ell}{2} |\vec{P}|^2 \right) \] 
\[ (3.107) \]

The coproducts of such operators are:

\[ \Delta(N_i) = N_i \otimes 1 + e^{-\ell P_0} \otimes N_i + \ell \epsilon_{ijk} P_j \otimes R_k \] 
\[ (3.108) \]
and all commutation relations except (3.100) are classical.

Boosts generators, like rotation ones, do not suffer from basis ambiguity; performing the previous analysis in another basis we would have found exactly the same expression for $N_i$. So the only generators to suffer from basis change ambiguity are the momenta, but as we will see with a proper choice of translation parameters is possible to make also translation transformation basis-independent.

3.2.4 Casimir operators

In constructing a scalar lagrangian we need a scalar operator in $\kappa$-Poincaré, i.e. one that commutes with with every other element of the algebra; we have to find the non-commutative deformation of the commutative mass casimir operator.

In Minkowski spacetime the kinetic part of the lagrangian for a scalar field is given by the mass operator:

\[ \Box = P_0^2 - |\vec{P}|^2 \] 
\[ (3.109) \]

that as its indices suggest commutes with every generator of the Poincaré algebra; it commutes with the momenta being a function only of them, and the commutators with the Lorentz sector give us:

\[ [R_i, \Box] = - \left( P_k \left[ R_i, P^k \right] + \left[ R_i, P^k \right] P_k \right) = 2i \epsilon_{ijk} P_k P_j = 0 \] 
\[ (3.110) \]

\[ [N_i, \Box] = P_0 \left[ N_i, P^0 \right] + \left[ N_i, P^0 \right] P_0 - P_k \left[ N_i, P^k \right] - \left[ N_i, P^k \right] P_k = i (P_0 P_i + P_i P_0 - P_j P_i - P_i P_j) = 0 \] 
\[ (3.111) \]
The deformation of the mass casimir in $\kappa$-Poincaré is the operator:
\[ \hat{\Box} = \left( \frac{2}{\ell} \right)^2 \sinh^2 \left( \frac{\ell P_0}{2} \right) - e^{\ell P_0} P_i^2 \quad (3.112) \]

that can be written in a more manageable way introducing the operators:

\[ \tilde{P}_0 = \frac{2}{\ell} \sinh \left( \frac{\ell P_0}{2} \right), \quad \tilde{P}_i = e^{\frac{\ell}{2} P_0} P_i \quad (3.113) \]

so that we have \[ \hat{\Box} = \tilde{P}_\mu \tilde{P}^\mu. \]

Using the defining commutation rules for the \(\kappa\)-Poincaré algebra we can easily show that this deformed mass operator is a casimir, i.e. commutes with all generators of the algebra. Also in this case it commutes with momenta because it’s a function of them; it’s easy to see that commutators with rotation generators - that maintain classical action - will give zero also in the presence of a deformation, since:

- \( \tilde{P}_0 \) is a function of \( P_0 \) only, so \( [R_i, P_0] = 0 \) \( \Rightarrow [R_i, \tilde{P}_0] = 0; \)
- \( e^{\ell P_0} \) commutes with \( R_i \), so \( [R_i, P_i P^i] = 0 \) \( \Rightarrow [R_i, \tilde{P}_i \tilde{P}^i] = 0. \)

Then again the only non trivial part is that of the boosts: for it we have to compute the commutators, exploiting the properties\(^{15}\) of \( \tilde{P}_\mu \). From them we get:

\[ [N_i, \tilde{P}_0] = i \cosh \left( \frac{\ell P_0}{2} \right) P_i \quad (3.114) \]

\[ [N_i, \tilde{P}_j] = i \left[ \delta_{ij} \left( 1 - \frac{e^{-2\ell P_0}}{2\ell} \right) - \frac{\ell}{2} |\vec{P}|^2 \right] \tilde{P}_i \tilde{P}_j \quad (3.115) \]

Applying these commutation rules we obtain:

\[ [N_i, \hat{\Box}] = [N_i, \tilde{P}_0^2] = [N_i, \tilde{P}_j^2] = \frac{i}{\ell} \left( e^{\ell P_0} - e^{-\ell P_0} \right) P_i \quad (3.116) \]

that imply, finally:

\[ [N_i, \hat{\Box}] = 0. \quad (3.117) \]

\(^{15}\)Derived in the appendix.
Chapter 4

κ-Minkowski spacetime symmetries and Field theory

In this chapter we want to give an idea of the usual approach to non-commutative spacetime physics, i.e. the definition of a field theory on the non-commutative coordinates, with the aim to gain an insight on the charges structure of the theory.

We start specifying that we are not trying to define a quantum field theory; our fields are classical, but defined on a non-commutative spacetime. This non-commutativity does not come from canonical commutation relations between fields, but is an a priori assumption of the theory. We don’t have, like in “commutative quantum field theory”, fields that are operator-valued distributions on coordinates; coordinates themselves are non-commuting operators acting on a pre-geometric space that we will define in the last chapter of the thesis, and fields become non-commutative only as functions of these non-commuting variables.

We shall see that for such “classical field theories” in non-commutative spacetime standard techniques, of course somewhat adapted to the context, can be fruitfully applied. In particular we can introduce a non-commutative action for our fields, and asking for it to be extremized we will find physical trajectories.

In the definition of a field theory we based our work on the exposition given in [14].

4.1 Commutative case

As we said in the last chapter, the lagrangian is a scalar function of the fields; so in order to contain a kinetic term it has to contain a casimir of the Poincaré algebra. The simplest example of lagrangian in the commutative case is the Klein-Gordon one, valid for a scalar field:

$$
\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi) (\partial^{\mu} \phi) - \frac{1}{2} m^2 \phi^2
$$

This lagrangian has the associated action:

$$
S(\phi) = \int d^4 x \mathcal{L} = \int d^4 x \left[ \frac{1}{2} (\partial_{\mu} \phi) (\partial^{\mu} \phi) - \frac{1}{2} m^2 \phi^2 \right]
$$
that is equivalent to:

\[ S'(\phi) = \int d^4x \left[ \frac{1}{2} \phi \partial_\mu \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 \right] \quad (4.3) \]

the first order variation of this actions produces the well known equation of motion:

\[ \left( \square + m^2 \right) \phi = 0 \quad (4.4) \]

Now we stress again one of the main properties of this equation - indeed one of the defining properties: its covariance under Poincaré transformations. Having constructed this equation only with scalar elements (both of the Poincaré algebra and of the Minkowski function algebra) we insured ourselves the validity of relativity axioms: if \( \phi \) is a solution for this equation, and \( G \) is an element of the Lorentz group, also \( G\phi \) is a solution. Explicitly:

\[ \left( \square + m^2 \right) G\phi = G \left( \square + m^2 \right) \phi + [\square, G] \phi = 0 \quad (4.5) \]

Where the last term vanishes due to the casimir nature of the \( \square \) operator.

### 4.2 Non-commutative case

In the last chapter we already found the non-commutative deformation for the mass casimir:

\[ \tilde{\square} = \tilde{P}_\mu \tilde{P}^\mu = \left( \frac{2}{\ell} \right)^2 \sinh^2 \left( \frac{\ell P_0}{2} \right) - e^{\ell P_0} p^2 \quad (4.6) \]

It is natural to construct a lagrangian with this casimir, that in the limit \( \ell \to 0 \) is the Klein-Gordon lagrangian, and to derive from it the non-commutative equations of motion. As we will see the associated equation is obviously a deformation of the classical one:

\[ \left( \tilde{\square} + m^2 \right) \phi = 0 \quad (4.7) \]

but already at this level we might have a complication. In the commutative case the request of covariance of the equations of motion is that if a field is a solution, so it is the Lorentz-trasformed one; in equations this mean:

\[ \left( \square + m^2 \right) e^{i\alpha^j G_j} \phi = 0 \quad (4.8) \]

with \( G_j \) arbitrary generators of Poincaré algebra.

Being the \( \alpha^j \) just real numbers they can be taken to the left of the l.h.s. (writing the power series of the exponential), so the covariance condition reduces to (4.5). In the non-commutative case, however, we will see that the transformation parameters may have non trivial properties, and in particular they can be more complex object than standard real numbers, a priori non commuting with the casimir of the algebra. Limiting ourselves to infinitesimal transformations, we have:
4.2 Non-commutative case

\[(\tilde{\Box} + m^2) \phi = 0 \implies (\tilde{\Box} + m^2) \left( \phi + i\alpha^j G_j \phi \right) = 0 \]
\[\implies (\tilde{\Box} + m^2) \alpha^j G_j \phi = 0 \]
\[\implies \alpha^j (\tilde{\Box} + m^2) G_j \phi + [\tilde{\Box}, \alpha^j] G_j \phi = 0 \quad (4.9)\]

In the next chapter we will follow previous works and assume that our parameters commute with translations generators, so we have \([\tilde{\Box}, \alpha^j] = 0\).

For now we can go on with our analysis, and assume that the non-commutative extension of the Klein-Gordon lagrangian is:

\[L = \frac{1}{2} \left[ (\tilde{P}_\mu \phi) (\tilde{P}^\mu \phi) - m^2 \phi^2 \right] \quad (4.10)\]

and the associated action:

\[S(\phi) = \frac{1}{2} \int d^4 \hat{x} \left[ (\tilde{P}_\mu \phi) (\tilde{P}^\mu \phi) - m^2 \phi^2 \right] \quad (4.11)\]

The first question we ask relatively to this action is if there exist a non-commutative analogue for the equivalence with the action¹:

\[S'(\phi) = \frac{1}{2} \int d^4 \hat{x} \left[ \phi \tilde{\Box} \phi + m^2 \phi^2 \right] \quad (4.12)\]

The answer to this question is not straightforward as in the commutative case because we don’t have a non-commutative analog of the divergence theorem. To prove the equivalence we use the coproduct rules² for \(\tilde{P}_\mu\) and \(e^{\ell P_0}\):

\[\Delta(\tilde{P}_\mu) = \tilde{P}_\mu \otimes e^{\ell P_0} + e^{-\ell P_0} \otimes \tilde{P}_\mu \quad (4.13)\]
\[\Delta(e^{\ell P_0}) = e^{\ell P_0} \otimes e^{\ell P_0} \quad (4.14)\]

to obtain:

\[\int d^4 \hat{x} \left[ (\tilde{P}_\mu \phi) (\tilde{P}^\mu \phi) \right] = \int d^4 \hat{x} \left[ \tilde{P}_\mu (\phi e^{-\frac{\ell}{2} P_0} \tilde{P}^\mu \phi) - e^{-\frac{\ell}{2} P_0} (\phi \tilde{P}_\mu \tilde{P}^\mu \phi) \right] = - \int d^4 \hat{x} \left[ \phi \tilde{\Box} \phi \right] \quad (4.15)\]

In the last equation we used two simple facts proved in the appendices:

\[\int d^4 \hat{x} \tilde{P}_\mu f(\hat{x}) = \int d^4 \hat{x} P_\mu f(\hat{x}) = 0 \quad (4.16)\]
\[\int d^4 \hat{x} \left( e^{\ell P_0} f(\hat{x}) \right) = \int d^4 \hat{x} f(\hat{x}) \quad (4.17)\]

¹This equivalence would make our calculations a lot easier.
²Derived in the appendices.
The last one, in particular, shows that under the integral sign we can treat \(\tilde{\Box}\) like \(\Box\) with respect to integration by parts:

\[
\int d^4 \hat{x} f(\tilde{\Box} g) = \int d^4 \hat{x} (\Box f) g
\]

(4.18)

Proved the equivalence of the two actions we can use either of them to derive the equation of motion; we will use the second one, as it simplifies our derivation (we will call it \(S\) instead of \(S'\)). As in the commutative case we perform a first order variation on the fields and ask for it to be ineffective on the action:

\[
\delta S = S(\phi + \delta \phi) - S(\phi) = \frac{1}{2} \int d^4 \hat{x} \left[ \delta \phi \tilde{\Box} \phi + \phi \tilde{\Box} \delta \phi + m^2 (\phi \delta \phi + \delta \phi \phi) \right] = 0
\]

(4.19)

To perform the next step we need another very useful fact about \(\kappa\)-Minkowski functions, the deformed cyclicity under integral sign:

\[
\int d^4 \hat{x} [fg] = \int d^4 \hat{x} (e^{-3\ell P_0} g) f = \int d^4 \hat{x} \left[ g(e^{3\ell P_0} f) \right]
\]

(4.20)

With this identity and integrating by part the term \(\phi \tilde{\Box} \delta \phi\), we can put the variation of the action in the form:

\[
\delta S = \frac{1}{2} \int d^4 \hat{x} \left[(1 + e^{-3\ell P_0})(\tilde{\Box} + m^2)\phi\right] \delta \phi = 0
\]

(4.21)

We can now go on in two distinct ways; one is to apply a “fundamental theorem of variational calculus” generalized to operators: to \(\delta S\) to be null with \(\delta \phi\) an arbitrary linear operator the rest of the integrand must be the null operator:

\[
(1 + e^{-3\ell P_0})(\tilde{\Box} + m^2)\phi = 0 \implies (\tilde{\Box} + m^2)\phi = 0
\]

(4.22)

where the implication arrow derive from the positive-definiteness of the operator \(e^{-3\ell P_0}\).

The other way, less “slippery” but more cumbersome, is to rely entirely on the Fourier transform version of the integral:

\[
\delta S = \frac{1}{2} \int d^4 \hat{k} \int d^4 k \int d^4 q \left[(1 + e^{-3\ell k_0})(\tilde{k}^2 - m^2)\phi(k)\right] \delta \phi(q) \Omega(e^{ik_\mu x^\mu}) \Omega(e^{ik_\mu x^\mu})
\]

\[
= \frac{1}{2} \int d^4 k \int d^4 q \left[(1 + e^{-3\ell k_0})(\tilde{k}^2 - m^2)\phi(k)\right] \delta \phi(q) \int d^4 \hat{\xi} \Omega(e^{ik_\mu x^\mu}) \Omega(e^{i\hat{\xi}_\mu x^\mu})
\]

\[
= \frac{1}{2} \int d^4 k \int d^4 q \left[(1 + e^{-3\ell k_0})(\tilde{k}^2 - m^2)\phi(k)\right] \delta \phi(q) \delta^4(k_\mu + e^{-\ell k_0}(1 - \delta^\mu_0) q_\mu)
\]

\[
= \frac{1}{2} \int d^4 k \left[(1 + e^{-3\ell k_0})(\tilde{k}^2 - m^2)\phi(k)\right] \delta \phi(-k_\mu e^{\ell k_0(1 - \delta^\mu_0)})
\]

(4.23)

The last line is a real integral, to which we can apply the “standard” fundamental theorem of calculus of variations to obtain:

---

See the appendices.
4.3 Deformed dispersion relations

Equation (4.4) in the commutative case poses a constraint on particles propagation in spacetime; it implies that a field has non null Fourier components only if those components satisfy the dispersion relation:

\[ p_0^2 - |\vec{p}|^2 = m^2 \]  

(4.28)

i.e. only if energy and momentum of a particle are in agreement with the laws of special relativity. From the dispersion relation we can define the group velocity of a particle:

\[ v_g = \frac{\partial p_0}{\partial |\vec{p}|} = \frac{\partial E}{\partial |\vec{p}|} = \frac{|\vec{p}|}{E} = \frac{|\vec{p}|}{\sqrt{\left|\vec{p}\right|^2 + m^2}} \]  

(4.29)

(of course if we wanted a single component of the velocity we would have to derive with respect to a single component of the momentum).

As we can see the velocity as a function of the mass has an absolute maximum for \( m = 0 \), with value 1 (equal to the speed of light in our units).

The generalization (4.25) to the non commutative case of this equation leads, if analyzed in the same way, to a striking result; first of all, it implies a less plain correlation between energy and spatial momentum:

\[ -\Box + m^2 = \left(\frac{2}{\ell}\right)^2 \sinh^2 \left(\frac{\ell k_0}{2}\right) - e^{\ell k_0} k_i^2 - m^2 = 0 \]

\[ \implies e^{\ell k_0} + e^{-\ell k_0} - 2 - \ell^2 e^{\ell k_0} k_i^2 - \ell^2 m^2 = 0 \]

\[ \implies e^{-2\ell k_0} - (2 + \ell^2 m^2) e^{-\ell k_0} + (1 - \ell^2 |\vec{k}|^2) = 0 \]

(4.30)

(4.31)

(4.32)

The last line is a quadratic equation in \( e^{-\ell k_0} \), that has solutions:

\[ e^{-\ell k_0} = \left(1 + \frac{\ell^2 m^2}{2}\right) \pm \sqrt{\left(1 + \frac{\ell^2 m^2}{2}\right)^2 - (1 - \ell^2 |\vec{k}|^2)} \]

\[ = \left(1 + \frac{\ell^2 m^2}{2}\right) \pm \ell \sqrt{(m^2 + |\vec{k}|^2) + \frac{\ell^2}{4} m^4} \]  

(4.33)
This solutions obviously reduce to the the classical ones in the limit $\ell \to 0$, but there are important differences in the case $\ell \neq 0$; we have:

$$k_0^\mp = -\frac{1}{\ell} \log \left( (1 + \frac{\ell^2}{2} m^2) \pm \ell \sqrt{(m^2 + |\vec{k}|^2) + \frac{\ell^2}{4} m^4} \right)$$  \hspace{1cm} (4.34)

We have written $k_0^\mp$ with reversed signs with respect to those in the logarithm because, as we can easily see:

$$\lim_{\ell \to 0} k_0^- = \sqrt{|\vec{k}|^2 + m^2}$$  \hspace{1cm} (4.35)

$$\lim_{\ell \to 0} k_0^+ = -\sqrt{|\vec{k}|^2 + m^2}$$  \hspace{1cm} (4.36)

that is, the non-commutative particle dispersion relation is that with the minus sign, the antiparticle one is that with the plus. These two solutions are connected by the relations:

$$e^{-\ell k_0^-} = (2 + \ell^2 m^2) - e^{-\ell k_0^+}$$  \hspace{1cm} (4.37)

$$k_0^- = -\frac{1}{\ell} \log \left( (2 + \ell^2 m^2) - e^{-\ell k_0^+} \right)$$  \hspace{1cm} (4.38)

In particular for the massless particles we have the following simplified relations:

$$k_0^\mp = -\frac{1}{\ell} \log(1 \pm \ell |\vec{k}|)$$  \hspace{1cm} (4.39)

$$k_0^- (e^{\ell k_0^+} |\vec{k}|) = -k_0^+ (|\vec{k}|)$$  \hspace{1cm} (4.40)

In the positive-energy solutions we can see both in the massive and in the massless case a strikingly new behaviour: instead of going to infinity in a linear way the energy has a vertical asymptote for $|\vec{p}| = \ell^{-1}$. This means that both the energy and its derivative with respect to the momentum go to infinity for momenta approaching the inverse of the characteristic length $\ell$. Being the derivative of the energy with respect to the momentum equal to the group velocity of a wave or to the speed of a particle (at least classically), we can deduce from this analysis the existence of massless particles propagating with a velocity different than $c$.

We have however to stay cautious with this analysis, in that we applied methods valid in the commutative case, as the association between classic fields and particles, that here seems to imply for example an asymmetry between particles and antiparticles.

### 4.4 Noether Analysis: Translations

Here we give an overview of the previous results obtained in the Noether’s analysis in $\kappa$-Minkowski space-time. Classically the idea is that if we manage to express the variation of the action (calculated on solutions of equation of motion) as the integral of a four-divergence, the current associated with this divergence will be the
4.4 Noether Analysis: Translations

conserved one. We start from the translation case, first analyzed in [23], and then switch to the Lorentz sector, analyzed in [24, 25]. In this section we will try to put over to the reader the ideas and intuitions that guided these previous works, though as we will see in the next sections some manipulations are "formal": the existence of some mathematical object we need is just supposed, and we will see that in some cases this supposition bring with it contradictions.

Classically we characterize a translation as the infinitesimal transformation associated to the (commutative) Lie algebra of the momentum generators:

\[ x^\mu \rightarrow x'^\mu = (1 + i \epsilon^\nu P_\nu) x^\mu = x^\mu + \epsilon^\mu \]  

(4.41)

where the \( \epsilon^\mu \) are real numbers small enough to the second order to be discarded.

To characterize the transformation on general (scalar) functions we have to specify if we are acting with an active or a passive transformation: in the first case the action is the same as the one on the coordinates:

\[ f(x) \rightarrow f(x') = f(x + \epsilon) \approx (1 + i \epsilon^\nu P_\nu) f(x) \]  

(4.42)

clearly only constant functions can be invariant under active translations (obviously to obtain a finite transformation we must exponentiate the generators). In the case of passive transformations however we can enforce, besides the argument variation, also a functional variation, with which we can make our function invariant under translations:

\[ f(x) \rightarrow f'(x') = \delta f(x) + (1 + i \epsilon^\nu P_\nu) f(x) \]  

(4.43)

The condition for the function to be invariant under changes of reference frame is given by:

\[ \delta f(x) = -df(x) \]  

(4.44)

with the \( d \) operator is the generator of translations \( i\epsilon^\mu P_\mu \).

4.4.1 Commutative case

We can now study the behaviour of our field theory under translation transformations; the action is defined as a spacetime integral of a scalar function (the lagrangian) of points in Minkowski spacetime so - being an integral over all spacetime - is invariant both under active and passive translations:

\[ dS = \int d^4x d\mathcal{L} = 0 \]  

(4.45)

\[ \delta S = \int d^4x [\delta \mathcal{L} + d\mathcal{L}] = 0 \]  

(4.46)

The first vanishes because it is the integral of a divergence, the second because has a vanishing integrand. We focus ourselves on passive transformations, being the two analysis very similar.

---

The reference [4] for the following statement: I.e. a function of spacetime points, rather than coordinates.
We can now try and express also the second integrand in the form of a four-divergence; we use here a free, massless lagrangian:

\[ \mathcal{L} = \phi \Box \phi \]  

\[ \delta S = \int d^4x [\delta \phi \Box \phi + \phi \Box \delta \phi + d\mathcal{L}] \]

and, with the aid of the condition \( \delta \phi = -d\phi \) and the equation of motion \( \Box \phi = 0 \), we obtain:

\[ \delta S = i \int d^4x P_\mu J^\mu \]  

\[ J^\mu = \epsilon^\mu \mathcal{L} - [\phi P^\mu (\epsilon^\rho P_\rho \phi) - (P^\mu \phi) (\epsilon^\rho P_\rho \phi)] \]

From the current \( J^\mu \) isolating the parameters of transformation, we obtain the standard energy-momentum conserved tensor.

### 4.4.2 Non-commutative free case

The next step is to try and extend the same procedure to the non-commutative case, allowing for the theory to tell us what to change in order to find a current.

We use the same definition of translations, except the fact that the \( \epsilon^\mu \) cannot be simple real numbers, but non-commutative objects, as can be seen from the condition that the translated algebra is still of \( \kappa \)-Minkowski type:

\[ [x^i + \epsilon^i, x^0 + \epsilon^0] = [x^i, x^0] + [\epsilon^i, \epsilon^0] + [\epsilon^i, x^0] + [\epsilon^0, x^0] = i\ell(x^i + \epsilon^i) \]

We point out here (and analyze more deeply later) the fact that the last factor \( i\ell \epsilon^i \) can arise from the last commutator \( [\epsilon^i, \epsilon^0] \), or from that between parameters and coordinates.

Let’s go on and apply a translation on our non-commutative action:

\[ S = \int d^4\hat{x}\phi \Box \phi \]

\[ \delta S = \int d^4\hat{x} [\delta \phi \Box \phi + \phi \Box \delta \phi + d\mathcal{L}] \]

\[ = \int d^4\hat{x} [\delta \phi \Box \phi + \phi \Box \delta \phi + i\epsilon^\mu P_\mu \mathcal{L}] \]

\[ = \int d^4\hat{x} [\delta \phi \Box \phi + \phi \Box \delta \phi + i\epsilon^\mu (P_\mu \phi) \Box \phi + (e^{-\ell P_0(1-\delta^0_0)} \phi) P_\mu \Box \phi)] \]

\[ = \int d^4\hat{x} [\phi \Box (-i\epsilon^\mu P_\mu) \phi + i\epsilon^\mu (e^{-\ell P_0(1-\delta^0_0)} \phi) P_\mu \Box \phi] \]  

As can be seen problems occurs already at the level of invariance of the action: if we let the translation parameters commute with the fields we obtain:\footnote{Assuming that the \( P_\mu \) commutes with themselves and with the \( \epsilon^\mu \).}
\[
\delta S = \int d^4x \left[ ie^\mu \left( e^{-\ell P_\mu (1-\delta^\nu_0)} - 1 \right) \phi \right] \hat{\Box} P_\mu \phi
\]

(4.54)

So, if we want an invariant action, the most natural assumption is that translation parameters and fields has non-zero commutator or, equivalently, that translation parameters commute with themselves but not with coordinates\(^6\).

In particular the commutator between fields and parameters must give precisely the same term given by the coproduct of translations generators:

\[
\phi e^\mu = e^\mu e^{-\ell P_\mu (1-\delta^\nu_0)} \phi \implies \delta S = 0
\]

(4.55)

We note two critical elements; first, when applied to single coordinates this commutator forces the \(\epsilon^\mu\) to belong to an algebra similar to the coordinates one:

\[
\left[ \epsilon^0, x^\mu \right] = 0 \quad \left[ \epsilon^\mu, x^i \right] = 0 \quad \left[ \epsilon^i, x^0 \right] = i\ell \epsilon^i
\]

(4.56)

second, if the parameters have this property, the operator \(d = ie^\mu P_\mu\) obey Leibniz rule:

\[
d(\phi \psi) = (ie^\mu P_\mu \phi)\psi + ie^\mu (e^{-\ell P_\mu (1-\delta^\nu_0)} \phi) P_\mu \psi =
\]

\[
= (ie^\mu P_\mu \phi)\psi + \phi (ie^\mu P_\mu \psi) =
\]

\[
= (d\phi) \psi + \phi (d\psi)
\]

(4.57)

If the parameters have this property then we have a conserved action, and can proceed in our Noether analysis:

\[
\delta S = \int d^4\hat{x} \left[ \delta \phi \hat{\Box} \phi + \phi \hat{\Box} \delta \phi + d\mathcal{L} \right]
\]

\[
= \int d^4\hat{x} \left[ \delta \phi \hat{\Box} \phi + \hat{P}_\mu \left( \left( e^{\ell P_0} \phi \right) \hat{P}^\mu \delta \phi \right) - e^{\ell P_0} \left( \left( \hat{P}_\mu \phi \right) \hat{P}^\mu \delta \phi \right) + d\mathcal{L} \right]
\]

\[
= \int d^4\hat{x} \left[ \delta \phi \hat{\Box} \phi + \hat{P}_\mu \left( \left( e^{\ell P_0} \phi \right) \hat{P}^\mu \delta \phi \right) - \left( e^{\ell P_0} \hat{P}^\mu \phi \right) e^{\ell P_0} \delta \phi \right] + e^{\ell P_0} \left( \left( \hat{\Box} \phi \right) \delta \phi \right) + d\mathcal{L}
\]

\[
= \int d^4\hat{x} \left[ P_\mu \left( \left( e^{\ell P_0} \phi \right) \hat{P}^\mu \delta \phi \right) - \left( e^{\ell P_0} \hat{P}^\mu \phi \right) e^{\ell P_0} \delta \phi \right]
\]

(4.58)

In the last passage we used, besides the equation of motion \(\hat{\Box} \phi = 0\), the identities\(^7\):

\[
\int d^4\hat{x} e^{\ell P_0} f(\hat{x}) = \int d^4\hat{x} f(\hat{x})
\]

(4.59)

\[
\int d^4\hat{x} \hat{P}_\mu f^\mu(\hat{x}) = \int d^4\hat{x} P_\mu f^\mu(\hat{x})
\]

(4.60)

Replacing now the \(\delta\) with the \(d\) operator we obtain:

\(^6\)We will deepen this issue later, but anticipate here that it’s the only reasonable option compatible with parameters-fields non-commutativity.

\(^7\)Proved in the appendices.
\[ \delta S = -i \int d^4 \tilde{x} P_\mu \left( \left( e \frac{P_0}{\sqrt{2}} \right) \tilde{P}^\mu e^\mu \phi - \left( e \frac{P_0}{\sqrt{2}} \right) e^\mu \tilde{P}_\mu \phi \right) \]
\[ = \int d^4 \tilde{x} P_\mu \dot{P}^\mu = 0 \] (4.61)

Interpreting the integrated current as the conserved one, and taking the spatial integral of its temporal component, we obtain a conserved charge, as can be verified with direct substitution of the solution of the equation of motion:
\[ P_0 \int d^3 \tilde{x} J^0 = 0 \] (4.62)

The next step could be to isolate the translation parameters to recover the standard energy-momenta tensor:
\[ e^\mu P_0 \int d^3 \tilde{x} T^0_\mu = 0 \Rightarrow P_0 \int d^3 \tilde{x} T^0_\mu = P_0 Q^P_\mu = 0 \] (4.63)

Doing so naively we obtain for the charges the expression:
\[ Q^P_\mu = \int d^3 \tilde{x} \left( \left( e^{-(1\frac{1}{2} - \delta^0_0)\ell P_0} \right) \tilde{P}^0 P_\mu \phi - \left( e^{\ell P_0} \tilde{P}^0 \phi \right) e^\mu \tilde{P}_\mu \phi \right) \] (4.64)

but, as we will see later, this step can’t be done without due reflections. We note in passing that the transformations we studied in this chapter forms a differential calculus that has been already studied by Oeckl in [26].

### 4.4.3 No physical ambiguity

As anticipated in previous chapters the only ambiguity in the definition of symmetry generators is in the spatial momentum basis, that depends on the choice of the Weyl map. This ambiguity must of course disappear somewhere before giving physical results, in that the choice of a particular Weyl map doesn’t correspond to anything physical. It can be argued that neither the action of a translation generator alone is something physical, because to physically translate a system we must act on the function representing this system with translation parameters as well with translation generators.

This intuition is indeed useful in our case, because we can prove the Weyl-basis ambiguity of the momenta to be absorbed by different translation parameters. As an example we try to define a translation in time-to-the-left basis that gives us a conserved charge; we recall that:
\[ \Delta(P^l_i) = P^l_i \otimes e^{\ell P_0} + 1 \otimes P^l_i \] (4.65)

and
\[ P^l_i = e^{\ell P_0} P^r_i \] (4.66)

Like we did in time-to-the-right basis, we have to impose the Leibniz property on spatial translations (the temporal one trivially satisfy it) to have a symmetry with conserved charge; if we write \( d_l = i e^{\ell P_0} P^l_i \) and try to enforce Leibniz rule we obtain:
\[ d_t(fg) = i\epsilon^i \left[(P_i^t f)e^{\epsilon P_b}g + fP_i^t g\right] \]
\[ = \left[(d_t f)e^{\epsilon P_b}g + fd_t g + i \left[\epsilon^i, f\right] P_i^t g\right] \] (4.67)

It’s clear that, being the two functions \( f \) and \( g \) arbitrary, and likewise arbitrary being the time dependence of \( g \), we can’t put the last line in the form we were looking for by an accurate choice of parameters. We can however change their order. We retain as translation parameters the ones defined in (4.56), so we have:

\[ d_r f = i\epsilon^i P_i^r f = i \left(e^{\epsilon P_b} P_i^r f\right) \epsilon^i = i \left(P_i^r f\right) \epsilon^i \] (4.68)

The object we want to be invariant is precisely the differential, that gives the function calculated in a transformed point, so we can take the last equation as a definition of translations in time-to-the-left basis. The generalization to the parametric basis is:

\[ df = i\epsilon^i P_i^r f = i \left(\epsilon^i\right)^\alpha \left(P_i^{(\alpha)} f\right) \left(\epsilon^i\right)^{1-\alpha} \] (4.69)

In this way we have a definition of differential which is invariant regardless of the basis we choose; changing the ordering relation between functions and parameters we can definitely reabsorb the different properties of various momentum-basis.

We remark that the only physical observables are the charge we want to derive in this Noether analysis, and that these charges are dependent only from differentials, not from momenta alone.

The last useful observation we can do about this removal of ambiguity in the differential is that the definition of, say, \( \alpha = \frac{1}{2} \) differential seems difficult to implement in the classical definition of exponential. We could argue that we need to exponentiate separately the parameters to the left and to the right of the translated function, but anyhow in this analysis the problem is not relevant, in that to look for a conserved charge we need only infinitesimal transformation.

### 4.5 Noether Analysis: Rotations

We saw in the second chapter that for rotations we can retain classical action and trivial coproducts, and that the form of (3.2) is left invariant without the necessity of a deformation.

The fact that the coproduct is trivial implies that the expression

\[ d = \sigma^i R_i \] (4.70)

is a good differential without the necessity of non-trivial properties for the \( \sigma^i \).

As we repeatedly noted the property of a symmetry operator of being a differential and to commute with the casimir is the only necessary condition to have a conserved charge, so choosing \( \sigma^i \) to be mere real numbers we respect all the hypothesis to have a charge.

We can repeat precisely the same path we followed for translations and obtain both invariance of the action and conserved charges:
\[ Q_i^R = \frac{1}{2} \int d^3x \left[ \left( e^{i\ell P_0} \phi \right) (\hat{P}_0 R_i \phi) - \left( e^{i\ell P_0} \tilde{P}_0 \phi \right) \left( e^{i\ell R_i} \phi \right) \right] \]  

(4.71)

Being here the transformation parameters just real numbers we took them out of the charge; we note moreover that given the triviality of parameters, both the differential \( d_R \phi = i\sigma^i R_i \phi \), and the transformed field \( R_i \phi \) are solutions of the equation of motion if \( \phi \) is a solution.

### 4.6 Noether Analysis: Boosts

For boosts we encounter a new level of conceptual and technical complication. The boost operators are not classical at all; they and their coproducts

\[ \Delta(N_i) = N_i \otimes 1 + e^{-\ell P_0} \otimes N_i + \ell \epsilon_{ijk} P_j \otimes R^k \]  

(4.72)

are a lot more cumbersome than translations and rotations ones.

There is in particular a difficulty of a whole new type: in the coproduct (4.72) don’t appear only boosts generators, but also translations and rotations one. This new feature implies that the equation imposing Leibniz rule on a boost transformation cannot be solved only in function of boosts parameters \( \tau^i \). As a matter of fact imposing this condition on the operator \( d_N = i\tau^i N_i \), acting on a product of functions we obtain:

\[ \left[ f(\hat{x}) \tau^i - \tau^i e^{-\ell P_0} f(\hat{x}) \right] N_i g(\hat{x}) = \ell \epsilon_{ijk} \tau^i P^j f(\hat{x}) R^k g(\hat{x}) \]  

(4.73)

We can see that the l.h.s. is independent from \( R_k g(\hat{x}) \), while the r.h.s depends from it. Given that the function \( g(\hat{x}) \) is arbitrary (and so it is its behaviour under boosts and rotations) we can not realize this identity without exotic actions of the terms containing \( \tau^i \) on \( N_i g(\hat{x}) \).

To get out from this impasse we execute a dimensional reduction; in 1+1 dimensions our situation is much less cumbersome: we do not have rotations, and the only Lorentz-sector generator is that of (a single) boost: we denote it \( N \). The coproduct of this generator in lower dimension can be shown to be:

\[ \Delta(N) = N \otimes 1 + e^{-\ell P_0} \otimes N \]  

(4.74)

This coproduct is the same we have for spatial momenta, so also the condition on parameters are the same in order to have Leibniz rule:

\[ [\hat{x}^0, \tau] = i\ell \tau \]  

(4.75)

Whatever relation the Lorentz parameters in 3+1 dimension have to observe, it must reduce to (4.75) in the 1+1-dimensional space-time.

We can ask ourselves, however, what are the parameters that has to reduce to \( \tau \) when two spatial dimension are discarded. One might intuitively say that the 3+1-dimensional correct analogue is:

\[ \tau N \rightarrow \tau^i N_i \]  

(4.76)
but this is misleading, in that this brings us an impossible condition on $\tau^i$. A moment of reflection will make us clear that in 1+1 dimensions $N$ is, by itself, all the Lorentz sector of the $\kappa$-Poincaré group, so it is natural the tentative extension:

$$\tau N \to \sigma^i R_i + \tau^j N_j$$ (4.77)

It is the operator above defined that reduces to $\tau N$ in lower dimensions. We can now try to enforce the Leibniz rule on the differential

$$d_L = i \left( \sigma^i R_i + \tau^j N_j \right)$$ (4.78)

What we obtain is:

$$\left( f(\hat{x})\tau^i - \tau^i e^{-\ell P_0 f(\hat{x})} \right) N_i g(\hat{x}) + \left( f(\hat{x})\sigma^k - \sigma^k f(\hat{x}) - \ell \epsilon_{jlk} \tau^j P^l f(\hat{x}) \right) R_k g(\hat{x}) = 0$$ (4.79)

This equation can be satisfied with the two different conditions:

$$f(\hat{x})\tau^i - \tau^i e^{\ell P_0} f(\hat{x}) = 0$$ (4.80)

$$f(\hat{x})\sigma^k - \sigma^k f(\hat{x}) = \ell \epsilon_{jlk} \tau^j P^l f(\hat{x})$$ (4.81)

We have thus managed to solve our problem combining rotations and boosts in such a way that the resulting operator is a derivation (complies with Leibniz rule). This implies, for boosts parameters, the same commutation rules seen for translations one:

$$\tau^i f(\hat{x}) = e^{-\ell P_0} f(\hat{x}) \tau^i \iff [\hat{x}^0, \tau^i] = i\ell \tau^i$$ (4.82)

The last expression is independent (in form) of the dimension, so it reduces straightforwardly to (4.75) in 1+1 dimensions.

For rotations we have a little more complicated condition:

$$[f(\hat{x}), \sigma_j] = \ell \epsilon_{jlk} \tau^l P^i f(\hat{x})$$ (4.83)

This condition can be worked out for single coordinates in the form:

$$[\sigma_j, \hat{x}_k] = i\ell \epsilon_{jlk} \tau^l , \quad [\hat{x}^0, \sigma^i] = 0$$ (4.84)

in a similar way to that used in the translation parameters case.

Given a set of parameters $\{\sigma^i, \tau^j\}$ satisfying the above-given conditions (and commuting with momenta), we have that the complete Lorentz differential

$$d_L = i \left( \sigma^i R_i + \tau^j N_j \right)$$ (4.85)

\*I.e. taking commutators of $\sigma^i$ with arbitrary powers of coordinates, then with plane waves, and finally extending by linearity to arbitrary functions.
satisfies all the conditions already stated to be a symmetry and to give to a conserved charge. This charge, as might be expected, is a linear combination of boosts and rotations ones:

\[ Q^L_j = \frac{1}{2} \int d^3x \left( \left(e^{-\ell P_0} \phi \right) \left( \bar{P}_0 N_j \phi \right) - \left( \bar{P}_0 \phi \right) \left(e^{\ell P_0} N_j \phi \right) \right) + + \epsilon_{jik} \left[ \left(e^{\ell P_0} P_l \phi \right) \left( \bar{P}_0 R_k \phi \right) - \left( e^{\ell P_0} \bar{P}_0 P_l \phi \right) \left(e^{\ell P_0} R_k \phi \right) \right] \]

We reported in this chapter all the Noether analysis made in previous works for completeness. This gives us a candidate conserved charge for every symmetry transformation in κ-Poincaré, and being us in the non-interacting case we can substitute solutions of equation of motion to check the invariance. All this procedure however will undergo a deep review in the next sections, both for enhance mathematical definition of the objects we used, and to find a procedure that we can apply also in the interacting case.

4.7 Equation of motion for interacting fields

As in the free case we define a lagrangian as a polynomial in fields and their derivative; in particular, as an example, we add to the kinetic part relative to the free fields a cubic power of the fields:

\[ \mathcal{L} = \phi \Box \phi + \phi^3 \]  
\[ S = \int d^4x \mathcal{L} = \int d^4x \left[ \phi \Box \phi + \phi^3 \right] \]

As it’s easy to see we have no obstructions in applying the same variational principle we used in the free case to this action, except the calculations are a little more involved; we have:

\[ \delta S = \int d^4x \left[ \delta \phi \Box \phi + \phi \Box \delta \phi + \phi^2 \delta \phi + \phi \delta \phi \phi + \delta \phi \phi^2 \right] = \]
\[ = \int d^4x \left[ \delta \phi \Box \phi - e^{\ell P_0} \bar{P}_u \left( \left(e^{\ell P_0} \bar{P}_0 \phi \right) \Box \delta \phi + e^{\ell P_0} \left( \left(\bar{P}_u \phi \right) \delta \phi \right) \right) \right] + + \left[ \phi^2 \delta \phi + \phi \delta \phi \phi + \delta \phi \phi^2 \right] = \]
\[ = \int d^4x \left[ \delta \phi \Box \phi - e^{\ell P_0} \bar{P}_u \left( \left(e^{\ell P_0} \bar{P}_0 \phi \right) \Box \delta \phi + e^{\ell P_0} \left( \left(\bar{P}_u \phi \right) \delta \phi \right) \right) \right] + + \left[ \phi^2 \delta \phi + \phi \delta \phi \phi + \delta \phi \phi^2 \right] = \]
\[ = \int d^4x \left[ \left(e^{-3P_0} \phi \right) \phi + \left( \bar{P}_0 \phi \right) \delta \phi + \phi^2 \delta \phi + \phi \delta \phi \phi + \delta \phi \phi^2 \right] = \]
\[ = \int d^4x \left[ \left(e^{-3P_0} \phi \right) \phi + \left( \bar{P}_0 \phi \right) \delta \phi + \phi^2 \delta \phi + \left(e^{-3P_0} \phi \right) \phi \delta \phi \right] + + \left[ \left(e^{-3P_0} \phi \right) \phi^2 \right] \delta \phi = \]
\[ = \int d^4x \left[ \left(1 + e^{-3P_0} \phi \right) \left( \phi^2 + \phi^2 \right) + \left(e^{-3P_0} \phi \right) \phi \right] \delta \phi = 0 \]  
(4.89)
Where we used as in other parts of the thesis the identities:

\[
\int d^4 \hat{x} fg = \int d^4 \hat{x} \left( e^{-3tP_0} g \right) f 
\]

\[
\int d^4 \hat{x} \tilde{P}_\mu f = \int d^4 \hat{x} P_\mu f = 0
\]

For (4.89) to hold for arbitrary \( \delta \phi \) the following equation of motion has to hold:

\[
\left( 1 + e^{-3tP_0} \right) \left( \tilde{\Box} \phi + \phi^2 \right) + \left( e^{-3tP_0} \phi \right) \phi = 0
\]

As we can see not only the derivation, but the equation of motion itself is a lot more involved than free-case one. For interacting fields we don’t have explicit solutions, so we cannot check, as in the last sections, if the charges associated to symmetries are conserved.

Looking carefully at this equation we see a further source of complications: unlike in the free case, here in the equation of motion are present terms that are explicit functions of the zeroth component of momentum alone. This might appear to inevitably spoil the covariance of this equation.

However the action of boosts in \( \kappa \)-Poincaré is highly nontrivial, and in other \( \kappa \)-Poincaré studies it was only thanks to such peculiar-looking factors that covariance was established. This is best known for the case of study of integration measure \[27\]. For what concerns specifically (4.92), the direct check of covariance would be however too troublesome, so we will skip it, being it not crucial for the subsequent development of the thesis.

### 4.8 Conserved charges for translation differential calculi

For ease of notation we will treat here a cubic interaction lagrangian (but as can be seen this procedure applies to every polynomial interaction):

\[
S = \int d^4 \hat{x} \mathcal{L} = \int d^4 \hat{x} \left[ \phi \tilde{\Box} \phi + \phi^3 \right]
\]

We will keep a formal definition of infinitesimal translations without specifying the properties of translation parameters, except that they have to commute with momenta (we assume them to be coordinate-independent, in order to be associated to translations); we will present again the distinction between active and passive transformations:

\[
\phi(\hat{x}') \approx (1 + i \epsilon^\mu P_\mu) \phi(\hat{x})
\]

\[
\phi'(\hat{x}') \approx \delta \phi(\hat{x}) + (1 + i \epsilon^\mu P_\mu) \phi(\hat{x})
\]

We now check the conservation of the action both for active and passive infinitesimal translations; in the first case we have:
\[ \delta S = \int d^4 \hat{x} [\mathcal{L}(\hat{x}') - \mathcal{L}(\hat{x})] = i \int d^4 \hat{x} \epsilon^\mu P_\mu \mathcal{L} \] (4.96)

and, given that \( \epsilon^\mu \) commutes with \( P_\mu \), we have simply:

\[ \delta S = \int d^4 \hat{x} P_\mu J^\mu = 0 \quad , \quad J^\mu = i \epsilon^\mu \mathcal{L} \] (4.97)

In the passive case again we find the necessity for nontrivial commutation rules between coordinates and translation parameters; being the lagrangian a scalar field the action is trivially conserved, being its variation the integral of the null total variation of the lagrangian. But to enforce the scalacity of the lagrangian we have

\[ \delta \mathcal{L} = -i \epsilon^\mu P_\mu \mathcal{L} \] (4.98)

and being the first operator \( \delta \mathcal{L} = \mathcal{L}(\phi + \delta \phi) - \mathcal{L}(\phi) \) a Leibniz operator, we have that also the r.h.s. of the equation must contain a Leibniz operator:

\[ i \epsilon^\mu P_\mu \phi \psi = d(\phi \psi) = \phi d\psi + (d\phi)\psi \] (4.99)

And this equation imply that the coproduct rule of \( \epsilon^\mu \) must balance out that of \( P_\mu \), that means, as we saw in the free case, for them to have the following commutation rules with coordinates\( ^9 \):

\[ [x^0, \epsilon^i] = i \ell \epsilon^i \quad [x^i, \epsilon^j] = [\epsilon^\mu, \epsilon^\nu] = [\epsilon^0, x^\mu] = 0 \] (4.100)

Now instead of performing a global analysis on the action, as in the free case, we try to apply the equation of motion directly to the lagrangian, as in the commutative case, in order to obtain a continuity equation for the current. We study only the case of active translations, being the two calculations essentially the same; the hypothesis we make for our theorem is the same we derived so far:

- trivial action of momenta on translation parameters: \( P_\mu \epsilon^\nu = 0 \);
- nontrivial commutation rule between temporal coordinate and spatial translation parameters.

This will make the operator \( d = i \epsilon^\mu P_\mu \) a point independent derivation, i.e. an operator satisfying Leibniz rule. Applying this operator to our lagrangian we obtain:

\[ d\mathcal{L} = (d\phi)\Box \phi + \phi \Box d\phi + \phi^2 d\phi + \phi (d\phi)\phi + (d\phi)\phi^2 \] (4.101)

To prove Noether’s theorem we want to put this equation in the form

\[ P_\mu J^\mu(\phi) = 0 \] (4.102)

when the fields in \( J^\mu \) are solutions of equation of motion.

The key passage to this result in the commutative case is to identify in the variation of lagrangian a term constrained to zero by the equation of motion, and to show that it is proportional to a divergence. We try to do the same in the

\( ^9 \)In the time-to-the-right basis.
non-commutative case. As a first step we move the \( \tilde{\square} \) operator in the r.h.s. of (4.101) from \( d\phi \) to \( \phi \) using coproduct rules of \( \tilde{P}_\mu \):

\begin{align}
\phi \tilde{\square} d\phi &= \tilde{P}_\mu [(e^{\ell P_0/2} \phi) \tilde{P}^\mu d\phi] - e^{\ell P_0/2} \tilde{P}_\mu \phi \tilde{P}^\mu d\phi \\
e^{\ell P_0/2} [\tilde{P}_\mu \phi \tilde{P}^\mu d\phi] &= e^{\ell P_0/2} \tilde{P}_\mu [(e^{\ell P_0/2} \phi) \tilde{P}^\mu d\phi] - e^{\ell P_0/2} [(\tilde{\square} \phi) d\phi]
\end{align}

So we obtain:

\[ dL = (d\phi) \tilde{\square} \phi + e^{\ell P_0/2} (\tilde{\square} \phi) d\phi + d\phi^3 + \tilde{P}_\mu \tilde{J}^\mu \]  

with

\[ \tilde{P}_\mu \tilde{J}^\mu = \tilde{P}_\mu [(e^{\ell P_0/2} \phi) \tilde{P}^\mu d\phi - (e^{\ell P_0} \phi) \tilde{P}^\mu e^{\ell P_0/2} d\phi] \]

This term is in the form of a divergence because all the components of \( \tilde{P}_\mu \) are at least of the first order in \( P_\mu \):

\[ \tilde{P}_0 = \frac{2}{\ell} \sinh \left( \frac{\ell P_0}{2} \right) = P_0 \sum_{n=0}^{\infty} \frac{(\ell P_0)^{2n}}{(2n+1)!} \]

\[ \tilde{P}_i = e^{\ell P_0/2} P_i = P_i e^{\ell P_0/2} \]

So we obtain \( \tilde{P}_\mu \tilde{J}^\mu = P_\mu J^\mu \) with the new current:

\[ J^0 = \sum_{n=0}^{\infty} \frac{(\ell P_0)^{2n}}{(2n+1)!} \tilde{J}^0, \quad J^i = e^{\ell P_0/2} \tilde{J}^i \]

Other terms that can be put in the form of a divergence is the higher orders in \( P_0 \) contained in \( e^{\ell P_0/2} (\tilde{\square} \phi) d\phi \):

\[ e^{\ell P_0/2} (\tilde{\square} \phi) d\phi = 1 + \sum_{n=1}^{\infty} \frac{(\ell P_0)^n}{n!} (\tilde{\square} \phi) d\phi \]

\[ = (\tilde{\square} \phi) d\phi + P_0 \left( \sum_{n=1}^{\infty} \frac{\ell^n P_0^{n-1}}{n!} (\tilde{\square} \phi) d\phi \right) \]

\[ = (\tilde{\square} \phi) d\phi + P_0 K^0 \]

where

\[ K^0 = \sum_{n=1}^{\infty} \frac{\ell^n P_0^{n-1}}{n!} (\tilde{\square} \phi) d\phi \]

So we can put (4.101) in the form:

\[ dL = d\phi \tilde{\square} \phi + (\tilde{\square} \phi) d\phi + d\phi^3 + P_\mu J^\mu \]

with now

\[ \text{Also in this part of the derivation we are ignoring the properties of covariance of exponential and formal power series, postponing this analysis to further works.} \]
\[ J^0 = \sum_{n=0}^{\infty} \frac{(\ell P_0)^{2n}}{(2n + 1)!} \tilde{J}^0 + K^0 \quad , \quad J^i = e^{\ell P_0} \tilde{J}^i \]  

(4.113)

In the commutative case the remaining terms not in form of a divergence would sum to zero when calculated on real trajectories, so we can say the current we found up to now is a “non-commutative deformation” of the classical one. The remaining parts will be due to the different behaviour of fields under global and local algebraic manipulations.

In particular in the derivation of the equation of motion we rely on commutator of fields under the integral sign, that brings us terms of type \(e^{-3\ell P_0}\); here we have to commute two fields in the same point of space, and it’s still unavailable the analytic expression for such a commutator. To find it we rely again on Fourier trasform, that gives us the cumbersome expression

\[ f(\hat{x})g(\hat{x}) = \exp \left( i\vec{x} \cdot [\vec{P}, e^{\ell P_0} - 1] \otimes \right) g(\hat{x})f(\hat{x}) \]  

(4.114)

where the index \( \otimes \) to the commutator is to indicate that the commutator has to be taken not respect to the algebra product, but to the tensor product:

\[[\vec{P}, e^{\ell P_0} - 1] \otimes g(\hat{x})f(\hat{x}) = \left( \vec{P}g(\hat{x}) \right) \left( (e^{\ell P_0} - 1)f(\hat{x}) \right) - \left( \vec{P} \leftrightarrow (e^{\ell P_0} - 1) \right) \]  

(4.115)

In spite of appearance this formula as a very nice property: it reduces to the global commutator \( (e^{-3\ell P_0}g)f \), plus divergence terms (the demonstration is quite involved, so we refer again to the appendix, where we also derive an explicit form for the current). With the aid of this property the theorem is almost proved; we can rewrite the variation of lagrangian (4.101) as:

\[ dL = dp\tilde{\square}\phi + (\tilde{\square}\phi)dp + d\phi\phi^2 + \phi d\phi\phi + \phi^2 d\phi + P_\mu J^\mu \]

\[ = \left( \mathbb{1} + e^{i\vec{x}[\vec{P}, e^{\ell P_0} - 1] \otimes} \right) (\tilde{\square}\phi + \phi^2) d\phi + \phi e^{i\vec{x}[\vec{P}, e^{\ell P_0} - 1] \otimes} d\phi + P_\mu J^\mu \]

\[ = \left( \mathbb{1} + e^{-3\ell P_0} \right) (\tilde{\square}\phi + \phi^2) + \phi e^{-3\ell P_0} \phi \right) d\phi + P_\mu J^\mu \]  

(4.116)

Finally we have a term constrained by the equation of motion, plus the divergence of the current:

\[ \tilde{J}^\mu = J^\mu + J^\mu_\text{local} \]  

(4.117)

which we rename, for compactness, again \( J^\mu \).

Now when \( \phi \) is a solution of the equation of motion we have \((\mathbb{1} + e^{-3\ell P_0})(\tilde{\square}\phi + \phi^2) + \phi e^{-3\ell P_0} \phi = 0\), and we are left with the equation:

\[ dL = P_\mu J^\mu \]  

(4.118)

As a last step we have to exploit again the property of parameters to commute with momenta, to obtain a divergence also in the l.h.s. of the equation:

\[ \text{See the appendix for the derivation.} \]
\[ d\mathcal{L} = ie^\mu P_\mu \mathcal{L} = P_\mu (ie^\mu \mathcal{L}) \] (4.119)

so, finally (with the last relabeling), we have the continuity equation:

\[ P_\mu J^\mu = 0 \] (4.120)

with

\[ J^0 = \sum_{n=0}^{\infty} \frac{(\ell P_0)^{2n}}{(2n + 1)!} \tilde{J}^0 + K^0 + J_{\text{local}}^0 - i\epsilon^0 \mathcal{L} \] (4.121)

\[ J^i = e^{\ell P_0} \tilde{J}^i + K^i + J_{\text{local}}^i - i\epsilon^i \mathcal{L} \] (4.122)

Also in this non-commutative spacetime equation of the type (4.120) are equivalent to charge conservation; for this equivalence we only need commutation of temporal momentum with spatial integration, and - by now well-known - properties of Fourier transform:

\[ P^\mu J_\mu = 0 \Rightarrow P_0 Q = P_0 \int d^3\hat{x} J^0 = \int d^3\hat{x} P_i J_i = 0 \] (4.123)

Equipped with this result, that gives us the explicit form of the conserved charge (in terms of the fields), we can infer that the results claimed by previous works about the free case are at least approximate; in the case of free fields we have:

\[ \mathcal{L} = \phi \Box \phi , \quad \Box \phi = 0 \] (4.124)

We can apply our new procedure to this lagrangian, at least to verify the conservation of the charge previously found.

With this lagrangian and equation of motion the second, third and fourth term in (4.121) are zero (on solutions), and we obtain the conservation equation:

\[ P_0 \int d^3\hat{x} \left( \sum_{n=0}^{\infty} \frac{(\ell P_0)^{2n}}{(2n + 1)!} \tilde{J}^0 \right) = 0 \] (4.125)

Assuming we can exchange integration and summation order (in this case both are well-behaved, fast-convergent mathematical objects) we can rewrite this equation as:

\[ P_0 \sum_{n=0}^{\infty} Q_n = 0 \] (4.126)

where

\[ Q_n = \frac{(\ell P_0)^{2n}}{(2n + 1)!} \tilde{J}^0 \] (4.127)

If this equation has to hold for each term of the sum alone (this is the case if we treat \( \ell \) as an arbitrary parameter) or we assume it is valid for the single term \( Q_0 \) (in this case we know that is true by previous works) then we obtain:
\[ Q_n = \frac{(\ell P_0)^2}{(2n + 1)} Q_{n-1} = 0 \quad \forall n \geq 1 \] (4.128)

So we have that [4.126] is redundant, and the physically sensible equation reduces to:

\[ P_0 Q_0 = 0 \] (4.129)

We obtain a more complicated current, but the same charge.

This is because higher order terms are all proportional to \( P_0 \); being the previous analysis \textit{global} they always appeared under an integral sign, and as we demonstrated:

\[ \int d^4 \hat{x} P_\mu f(\hat{x}) = 0 \] (4.130)

This won’t be the main correction to the previous analysis because, as we will see, the two hypothesis at page 56 are just too much to ask for the trasformation parameters, even in the simple case of translations.

As a final remark we note again that the only sufficient conditions to have Noether’s theorem is the existence of infinitesimal transformation operators\(^\text{12}\) that give rise to a differential calculus, i.e. that all respects Leibniz rule, and commutes with the casimir of the \( \kappa \)-Poincaré group. In case of infinitesimal transformations we can reverse our demonstration to show that they’re also necessary conditions. It’s worth to note that alternatives to our differential calculus exists, and we try to analyze them in the following.

### 4.9 Aside on different types of translation parameters

We want to focus again on the crucial properties we want our transformation parameters to respect. We saw in the last chapters that the main hypothesis used in the previous works to ensure the presence of a conserved charge are commutativity between momenta and infinitesimal symmetry transformations and Leibniz rule. We have seen moreover that under these hypothesis a non-commutative analogous of Noether’s theorem can be found to apply to free and interacting theories.

In this section we want to see all possible candidates of symmetry transformations that comply with these conditions (at least for translations). A naive characterization of translations, at least in classical theories, is simply to “add constant quantities to coordinates”. The “constant quantities” are the translation parameters:

\[ x^\mu + \epsilon^\mu = (1 + i \epsilon^\nu P_\nu) x^\mu \] (4.131)

From this characterization, asking only for the conservation of the \( \kappa \)-Minkowski defining commutation relations [3.2], we obtain two possible sets of parameters. We impose:

\[ [\hat{x}^i + \epsilon^i, \hat{x}^0 + \epsilon^0] = i \ell (\hat{x}^i + \epsilon^i) \] (4.132)

\(^{12}\)Here these operators are the combinations \( e^\mu P_\mu \) with \( e^\mu \) our non-commutative translation parameters.
Exploiting trivial properties of commutators we have:

\[ [\hat{x}^i, \hat{x}^0] + [\epsilon^i, \hat{x}^0] + [\hat{x}^i, \epsilon^0] = i\ell (\hat{x}^i + \epsilon^i) \quad (4.133) \]

Now the first term on l.h.s. cancels out the first term on r.h.s, leaving us with a condition on the transformation parameters:

\[ [\epsilon^0, \hat{x}^i] + [\hat{x}^i, \epsilon^0] + [\epsilon^0, \epsilon^i] = i\ell \epsilon^i \quad (4.134) \]

We can choose between parameters that commute with coordinates but don’t commute with themselves, giving a second copy of the $\kappa$-Minkowski algebra:

\[ [\epsilon^i, \epsilon^0] = i\ell \epsilon^i \quad (4.135) \]

A second possibility is a set of parameters commuting between themselves, but not commuting with coordinates:

\[ [\epsilon^0, \hat{x}^i] + [\hat{x}^i, \epsilon^0] = i\ell \epsilon^i \quad (4.136) \]

This possibility is very appealing for our approach, because allow us to perform transformations with arbitrary parameters; in the l.h.s. of the equation we have the commutator $[\epsilon^0, \hat{x}^i]$ that has no counterpart in the r.h.s.: multiplying $\epsilon^0$ by a real number the commutation relations would be spoiled, so we can assume

\[ [\epsilon^0, \hat{x}^i] = 0 \quad (4.137) \]

As a byproduct of this assumption we have also

\[ [\epsilon^i, \hat{x}^0] = i\ell \epsilon^i \quad (4.138) \]

These are precisely the commutation relations needed for parameters to ensure that translations comply with Leibniz rule.

The first possibility (4.135), on the other hand, poses stronger constraints on parameters: we can’t retain the possibility to multiply $\epsilon^0$ by arbitrary real numbers if we have $\epsilon^i \neq 0$, in that this would spoil the commutation relations. We must have fixed time-translation parameter and free spatial-translation ones, or free time translations but no spatial translations at all. Another drawback of this set of parameters is that, commuting them trivially with coordinates, they can not be used to impose Leibniz rule on translations, and so they can not form a differential calculus.

In [27] authors introduced an alternative four dimensional differential calculus, which features have still to be analyzed.

Another, more involved way to have a differential calculus has been worked out by Sitarz and others in [28, 21, 29, 30]. In this papers the possibility of using a five dimensional differential calculus is analyzed: the differential are called $dx^\mu, \phi$, with as usual $\mu \in 0, 1, 2, 3$ and $dx^4 = \phi$.

The commutation relations this calculus has to comply with are:

\[ [\epsilon^i, \hat{x}^0] = i\ell \epsilon^i \quad (4.138) \]

---

13The very presence of this possibility for translation parameters is what makes $\kappa$-Minkowski the homogeneous space for $\kappa$-Poincaré.

14Here we change the notation from $\hat{x}$ to $x$ for brevity.
\[
[x_\mu, \phi] = dx_\mu \\
[x_0, dx_0] = \ell^2 \phi \\
[x_i, x_j] = \delta_{ij} \ell (dx_0 - \ell \phi) \\
[x_0, dx_i] = 0 \\
[x_i, dx_0] = \ell dx_i
\] (4.139) (4.140) (4.141) (4.142) (4.143)

Using the variables \( dx^\mu \) as our translation parameters \( \epsilon^\mu \) we have a transformation operator:

\[
d_{5d} = dx^\mu P_\mu + \phi P_4
\] (4.144)

with \( P_4 \) is the operator:

\[
P_4 = \frac{\ell}{\ell} \left( \cosh(\ell P_0) - \frac{\ell^2}{2} e^{\ell P_0} |\vec{P}|^2 - 1 \right)
\] (4.145)

With the above-defined translation operator we have Leibniz rule for translations, and in the free case five conserved charges have been found corresponding to the five different translation generators (of course only four of them are non-trivial).

We mention here the five dimensional differential calculus only to remark that its features make it another possible candidate for our Noether’s theorem generalization, in that it seems to verify all the needed hypothesis. In the following we will focus only on the four dimensional candidate differential calculus, and leave the five dimensional one to future works.

### 4.10 Representations for coordinates and trasformation parameters

We have seen that an infinitesimal translation acts quite trivially on single coordinates; it simply adds the parameters of transformation to the coordinates:

\[
(1 + ie^\mu P_\mu) \hat{x}^\nu = \hat{x}^\nu + \epsilon^\nu
\] (4.146)

So, if we assume the \( x^\mu \) to be non commutative operators on an Hilbert space, the \( \epsilon^\mu \) have to be of the same kind, in order for the sum to have identical properties. In particular this implies, as we assumed so far, that we can treat them as elements of the same algebra of the coordinates, and take arbitrary products of them with generic functions. This simple fact, however, brings with it some contradictions: assuming the commutation relations necessary to have Leibniz rule

\[
e^i f(\hat{x}) = e^{iP_0} f(\hat{x}) e^i
\] (4.147)

we have problems, for example, with triple products of parameters and functions:

\[
f(\hat{x}) \epsilon^i g(\hat{x}) = e^i \left( e^{-iP_0} f(\hat{x}) \right) g(\hat{x})
\] (4.148)
but, considering $\epsilon^i g(\hat{x})$ as a new function $(\epsilon^i g)(\hat{x})$, we have:

$$f(\hat{x})\epsilon^i g(\hat{x}) = \exp\left(i\vec{x} \cdot [\vec{P}, e^{-\ell\hat{P}_0} - 1]_{\otimes} \right) (\epsilon^i g(\hat{x})) (f(\hat{x}))$$  \hspace{1cm} (4.149)$$

and if we assume the spatial parameters to commute with momenta and spatial coordinates we can take it to the left:

$$f(\hat{x})\epsilon^i g(\hat{x}) = \epsilon^i \exp\left(i\vec{x} \cdot [\vec{P}, e^{-\ell\hat{P}_0} - 1]_{\otimes} \right) g(\hat{x}) f(\hat{x})$$  \hspace{1cm} (4.150)$$

obtaining:

$$f(\hat{x})\epsilon^i g(\hat{x}) = \epsilon^i f(\hat{x}) g(\hat{x})$$  \hspace{1cm} (4.151)$$

That is in evident contradiction with (4.148).

To solve the impasse comes in our help some representation theory; in recent years a number of articles [31, 32] answered the question of what are the features of our $\kappa$-Minkowski coordinates when expressed as Hilbert-space operators. We report here the crucial features of this analysis, not dwelling too much on mathematical details for reasons of space.

Taking the spectrum of the operator:

$$R = \sqrt{\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2}$$  \hspace{1cm} (4.152)$$

we see that it contains only positive real values, so for its spectral representation we have:

$$R = \int_0^\infty rE(r)dr$$  \hspace{1cm} (4.153)$$

where $E(r)$ is the orthogonal projector on the $r$-eigenspace.

Now taking the projector on the strictly positive $r$-space:

$$E(r > 0) = \int_0^\infty E(r)dr - E(0) = \chi(0,\infty)(R)$$  \hspace{1cm} (4.154)$$

we can argue that this operator commutes with all generators of $\kappa$-Minkowski Lie algebra; it trivially commutes with all spatial coordinates due to the (3.2), and from the commutation formula for exponentials:

$$e^{ik_i \hat{x}_i} e^{-ik_0 \hat{x}_0} = e^{-ik_0 \hat{x}_0} e^{ik_0 k_i \hat{x}_i}$$  \hspace{1cm} (4.155)$$

$$e^{ik_0 \hat{x}_0} e^{ik_i \hat{x}_i} e^{-ik_0 \hat{x}_0} = e^{ik_0 k_i \hat{x}_i}$$  \hspace{1cm} (4.156)$$

taking an infinitesimal, arbitrary $k_i$ in the central exponential in the l.h.s. we obtain a relation for the $\hat{x}_i$:

$$e^{ik_0 \hat{x}_0} \hat{x}_i e^{-ik_0 \hat{x}_0} = e^{-\ell k_0} \hat{x}_i$$  \hspace{1cm} (4.157)$$

that in terms of the operator (4.154) becomes:

\footnote{We rely again on functional calculus.}
\[ e^{ik_0 \hat{x}^0} \chi(0,\infty)(R) e^{-ik_0 \hat{x}^0} = \chi(0,\infty)(e^{-\ell k_0}R) = \chi(0,\infty)(R) \] (4.158)

The last relation in turn implies that \( \chi(0,\infty)(R) \) commutes with \( \hat{x}^0 \), and so with all the generators of \( \kappa \)-Minkowski algebra.

Having an operator that commutes with all the generators of a Lie algebra (and assuming the representation in which they live is irreducible) we can apply the (generalized version of) Schur’s lemma: we can conclude that \( \chi(0,\infty)(R) \) is either the identity or the null operator.

We have reached a truly significant physical result; we have obtained that every representation of \( \kappa \)-Minkowski algebra falls in one of two classes: representations that contain the origin of spatial coordinates, and representations that do not. The latter case is somewhat trivial, in that we have all the \( \hat{x}^i \) equal to zero, and we can take \( \hat{x}^0 \) diagonal, with spectrum \( \mathbb{R} \).

The former instead is the more involved, and physically interesting; given the fact that the zero eigenvalue is not in the spectrum of \( R \) we can define the operator \( R^{-1} \), and the “angular variables” \( c^i = R^{-1} \hat{x}^i \). Now an interesting fact happens: we can study the commutators of these angular variables with coordinates, and we find that

- they commute with spatial coordinates, like \( R \) do;
- they commute with time coordinate also, because we have

\[ e^{\hat{x}^i}e^{-\hat{x}^i} = e^{\hat{x}^i}R^{-1}e^{-\hat{x}^i}e^{\hat{x}^i}e^{-\hat{x}^i} = (e^{\hat{x}^i}R^{-1})(e^{-\hat{x}^i}R^i) = c^i \] (4.159)

So, in the end, we can apply to them the same conclusions we have drawn for \( \chi(0,\infty)(R) \): they can be only zero or multiples of the identity operator. We take in consideration only the latter case, being the former of the same type of \( R = 0 \).

Being the \( c^i \) variables of commutative types, we can write the \( \kappa \)-Minkowski algebra in the new form:

\[ [\hat{x}^0, R] = i\ell R \ , \ \hat{x}^i = c^i R \] (4.160)

where all the nontrivial structure of the spatial coordinates is in \( R \).

In this way we have proved that all representations of the \( \kappa \)-Minkowski algebra in \( (3 + 1) \) dimensions are of type (4.160), with a different, irreducible representation for each different vector \( \vec{c} \). In this notation to the vector \( \vec{c} = \vec{0} \) we associate the trivial representation \( R = 0 \). All these irreducible representation, however, are not all independent; as one could think with a comparison with the commutative case all the independent \( \vec{c} \) can be taken to be elements of a unit sphere. To show this, we consider an arbitrary irreducible representation of type (4.160), with \( |\vec{c}| = \sqrt{c_1^2 + c_2^2 + c_3^2} \); from (4.157) we have:

\[ e^{i\frac{\log|\vec{c}|}{\ell} \hat{x}^i}e^{-i\frac{\log|\vec{c}|}{\ell} \hat{x}^i} = \frac{\hat{x}^i}{|\vec{c}|} \] (4.161)
and taking as new coordinates $\hat{x}^i = \hat{\tilde{x}}^i$, we have also $|\tilde{c}| = 1$.

With observation concerning representation theory in $\kappa$-Minkowski we can derive the explicit form of the translation parameters, starting from their commutation rules. Indeed, looking to them:

$$[\epsilon^i, \hat{x}^0] = i\ell \epsilon^i$$  \hspace{1cm} (4.162)

we can see at a glance that they are the same as that of spatial coordinates; the set $\{\hat{x}^0, \epsilon^i\}$ forms another copy of the $\kappa$-Minkowski algebra, and is in the same representation of the coordinates, since they act on the same vector space. Another way of seeing this fact is that the operators $(\hat{x}^0, \hat{x}^i, \epsilon^i)$ form a seven-dimensional $\kappa$-Minkowski algebra (the operator $\epsilon^0$, commuting with everything, is trivial, i.e. proportional to identity); from this outlook we conclude that:

$$[\hat{x}^0, R] = i\ell R, \quad \hat{x}^i = c^i R, \quad \epsilon^j = d^j R$$  \hspace{1cm} (4.163)

and, since $c^i$ and $d^j$ are just real numbers:

$$\epsilon^i = c^i \hat{x}^j$$  \hspace{1cm} (4.164)

where the two-indexed object $\epsilon^i_j$ is a 3x3 matrix expressing the most general linear dependence between parameters and coordinates.

We have thus proved that the two previously assumed hypothesis on translation parameters are not compatible; the commutation rules necessary to assure the Leibniz rule (and so the invariance of the action) imply a linear dependence of these parameters from spatial coordinates, in the sense of (4.164). This poses a serious threat on the content of the previous chapter, in that the basis of some demonstration derive from the last theorem. In the next chapter instead we will give a more compelling alternative the this irreducible $\kappa$-Minkowski representation: we will show that the natural setting to represent $\kappa$-Minkowski coordinates is phase space, and in that setting this limitation will be dropped out and a more natural way of representing translation parameters will be shown.

### 4.11 Conserved charges for free and interacting fields

In the derivation of charge conservation, to the continuity equation only two steps are sensitive to the space dependence of the parameters; the first is in the kinetic term, when we write:

$$d(\phi \hat{\Box} \phi) = d\phi \hat{\Box} \phi + \phi \hat{\Box} d\phi$$  \hspace{1cm} (4.165)

and the second is in translating the differential of the lagrangian in a divergence:

$$d\mathcal{L} = P_\mu J^\mu$$  \hspace{1cm} (4.166)

these two passages can not be performed if there is some coordinate dependence in the parameters of transformation; in the next two sections we analyze the

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\[16\] That we assume to be irreducible.
consequences of this complications for the Oeckl calculus, and then we will pause to reflect on the coordinate dependence of boosts parameters.

4.11.1 Oeckl transformations

The equation (4.165) is clearly wrong if in the $d$ operator there is some coordinate dependence, as in our case: $d = ie^\mu P_\mu$, $e^i = e^i_j \hat{x}^j$.

We have to calculate the last term taking into consideration this dependence:

\[
\hat{\Box} d\phi = i\hat{\Box} e^\mu P_\mu \phi = i\hat{P}^\rho [(\hat{P}_\rho e^\mu) e^{\ell P_0} P_\mu \phi + (e - \ell P_0) \hat{P}_\rho P_\mu \phi] = i\hat{P}^\rho [-i\delta^\rho_\mu e^{\ell P_0} P_\mu \phi + \epsilon^\mu \hat{P}_\rho P_\mu \phi] \quad (4.167)
\]

where in the last equality we used the form of the parameters derived in the last chapter, and their independence from time.

Applying again $\hat{P}_\mu$ we get:

\[
\hat{\Box} d\phi = i \left[ -i2\delta^\rho_\nu e^{\ell P_0} P_\nu P_i \phi + \epsilon^\mu \hat{\Box} P_\mu \phi \right] \quad (4.168)
\]

In the last expression the space dependence of $\epsilon^\mu$ generates the term $2\delta^\rho_\nu e^{\ell P_0} P_\nu P_i \phi$, which is not in the form of a divergence, and so we must take it into account in the discussion of the charge conservation.

The second step where we have to be cautious about dependence on coordinates is the transformation of the differential of the lagrangian in a divergence:

\[
dL = i\epsilon^\mu P_\mu L = P_\mu (i\epsilon^\mu L) \quad (4.169)
\]

again, this equality does not hold if $e^\mu$ has some space dependence. We can however easily find the correct modification; thanks to the coproduct rule of $P_\mu$ we have:

\[
P_\mu (i\epsilon^\mu L) = i (P_\mu \epsilon^\mu) L + i \left( e^{-\ell P_0} \epsilon^\mu \right) P_\mu L = \sum_{i=1}^3 \epsilon^i L + i\epsilon^\mu P_\mu L \quad (4.170)
\]

We have thus the two modified identities to put in the demonstration of chapter 5:

\[
\phi d\hat{\Box} \phi = \phi \hat{\Box} d\phi + 2\epsilon^i_j \phi e^{\ell P_0} P^j P_i \phi \quad (4.171)
\]

\[
dL = P_\mu (i\epsilon^\mu L) - \sum_{i=1}^3 \epsilon^i L \quad (4.172)
\]

These two additional terms give us an “almost” continuity equation:
\[ P_\mu J^\mu = \sum_{i=1}^{3} \epsilon^i_\mu \mathcal{L} - 2 \epsilon^j_\mu \phi \epsilon^{Pl} P^j P_i \phi \] (4.173)

There is no reason to the r.h.s. of the equation to give a null integral over all space, so we obtain

\[ P_0 Q = f(\epsilon^i_j, \phi) \] (4.174)

So we see that the only way to obtain a conserved charge - and, at the same time, to make the transformation a symmetry - is to choose a matrix \( \epsilon^i_j \) so that \( f(\epsilon^i_j, \phi) = 0 \).

An obvious way to achieve the result is to pick \( \epsilon^i_j \) so that:

\[ \sum_{i=1}^{3} \epsilon^i_i = 0 \quad \text{and} \quad \epsilon^i_j + \epsilon^j_i = 0 \] (4.175)

So that our parameters must be antisymmetric matrices in the coordinates, to generate a transformation of the type:

\[ d = i\alpha^i \epsilon_{ijk} x^j P^k \] (4.176)

i.e. they correspond to the generators of rotations.

We reached an odd result: we started thinking we were analyzing translation transformations and their properties in terms of conserved charges. Our new mathematical formalism forced us to conclude that, if we want a symmetry and a conserved charge, the transformations we are using reduce to rotations.

This is a truly deep, unsettling conceptual result: we ruled out translations from this differential calculus, and have to look for other candidates to which apply our formalism.

### 4.11.2 Lorentz sector

The next step could be to analyze the Lorentz sector, for which we have again conserved charges in the free field theory. We aspect this analysis to be much more cumbersome than that necessary for translations, so we will postpone it to other studies, but make some observation to clarify that also in that case the hypothesis used in previous works can be contradictory.

We have seen the hypothesis the Lorentz parameters have to comply to have a charge associated to boosts are:

\[ [\hat{x}^0, \tau^i] = i\ell \tau^i \quad , \quad [\hat{x}^j, \tau^i] = 0 \] (4.177)

\[ [\sigma_i, \hat{x}^j] = i\ell \epsilon_{ijk} \tau^k \quad , \quad [\sigma_i, \hat{x}^0] = 0 \] (4.178)

with \( \tau \) and \( \sigma \) boost and rotation parameter respectively.

It’s apparent that the commutation relations for boost parameters are the same of translation’s ones, so we can conclude that also the \( \tau \) have a linear dependence from coordinates:
\[ \tau^i = t^i_j \hat{x}^j \quad (4.179) \]

This implies that the first equation of \((4.178)\) becomes:

\[ [\sigma_i, \hat{x}_j] = i\ell\epsilon_{ijk} t^k_l \hat{x}^l \quad (4.180) \]

Applying a momentum operator to the last equation we obtain:

\[ [(P_\mu \sigma^i), \hat{x}^j] + (e^{-\ell P_0} - 1) \sigma^i \delta^j_\mu = i\epsilon_{ijk} t^k_l \delta^j_\mu \quad (4.181) \]

So, separating time and spatial components:

\[ [(P_l \sigma^i), \hat{x}_j] = i\epsilon_{ijk} t^k_l \quad \forall l \in \{1, 2, 3\} \quad (4.182) \]

\[ (e^{-\ell P_0} - 1) \sigma^i = 0 \quad (4.183) \]

The first of this equation is the most significant for us: it implies, if we want a non-zero boost transformation, to have a space-dependent rotation parameter (otherwise the commutator would be zero).

The condition of space-dependence of rotation generators is somewhat conflicting with the null commutator with time coordinate. We derived in the appendices the identity:

\[ [\hat{x}^0, f(\hat{x})] = -\ell \vec{x} \cdot \vec{P} f(\hat{x}) \quad (4.184) \]

This identity, applied to \(\sigma^i\), gives the quite tight condition:

\[ [\hat{x}^0, \sigma^i] = -\ell \vec{x} \cdot \vec{P} \sigma^i = 0 \quad (4.185) \]

that leaves us with little freedom, if any, to choose a space dependence for rotation parameters. We see thus that also in the Lorentz sector we have to be careful with our assumptions, and we have to verify the existence of a differential calculus complying with the hypothesis needed to apply our Noether’s analysis.

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4. \(\kappa\)-Minkowski spacetime symmetries and Field theory
Part III

Quantum particles on
non-commutative spacetime
Chapter 5

Representation of non-commutative spacetime

In the last chapter we saw how in the field theory treatment of a non-commutative spacetime, if we choose an irreducible representation for coordinates the resulting physical picture is that of a non-covariant space. First, the origin of space acquires a very special role in that it lives in a representation of its own; second, we cannot find room for a satisfactory representation of translation transformation. In particular it seems impossible to represent translation parameters from which derive a Noether translation charge.

To come out from this impasse we will exploit the formalism introduced in the second chapter of this thesis. In a covariant formulation of quantum mechanics we have a kinematical space in which the four coordinates are operators commuting between themselves. This is a perfect theory to deform, in order to obtain a representation of non-commutative coordinate operators.

We will perform the deformation assuming the spacetime coordinates $x^\mu$ on the kinematical space to be non-trivial functions of the auxiliary, commutative coordinates $q^\mu$ on which the kinematical space itself is defined and of the conjugate momentum $p_\mu$. We will choose these functions in order to obtain the commutation relations we want to represent and the right commutative limit. Given the role played by the kinematical space in the definition of the physical space, such deformations will in general (but not always, as we will see) reflect in deformations of the physical observables, and in turn to experimental predictions.

An alternative way to proceed could be to deform some coordinate observables directly on the physical Hilbert space; however the coordinate observables which don’t spoil the Lorentz symmetry which we want to deform are the $\chi^\mu$, which don’t commute already in the $\ell = 0$ case. On the other hand the commutation rules which we try to represent have the form:

$$[x^\mu, x^\nu] = i\ell \Upsilon^{\mu\nu}(x, p)$$ (5.1)

and the right hand side of the last equation goes to zero when $\ell \to 0$. So in the $\ell \to 0$ limit we need commutative coordinate operators, and this fact rules out the vector physical coordinates $\chi^\mu$. 

The deformation of kinematical coordinates in order to obtain the right commutation rules is an hint on how we have to deform the Hilbert space structure, and in particular the scalar product. First, we want the deformed kinematical coordinates to remain hermitian operators, so as a consequence we will have to deform the scalar product as well\footnote{Actually we will see that in a a particular representation choice we can use non hermitian coordinate operators, though this non hermiticity will not have any physical consequence.}. Second, and more important, the deformed commutation relations, to be covariant, imply a deformation of the symmetry group of the theory. we want of course the scalar product to remain invariant under the same deformed symmetry group, so a deformation of the scalar product will be necessary.

Of course the reasoning could be reversed: we could start deforming the scalar product, and then look for the new hermitian coordinates complying with deformed commutation relations. Though in the practical cases the starting point will be the deformation of coordinates, in this chapter I want to give a more geometrical account of the process, starting from the possible deformations of the scalar product, and deriving the possible compatible deformation of coordinates. In such a way it will be clear the primary role played by symmetries in the deformation of spacetime.

From now I will drop the 'hat' symbol to indicate operators, in that it will be clear when I will be speaking of operators and when of their eigenvalues.

### 5.1 Scalar product

In Minkowski spacetime we choose a scalar product in the kinematical space which is invariant under the Poincaré symmetry group. To realize this condition we need obviously an integration measure which is a scalar under a Lorentz transformation. This invariant is the volume form in momentum space (or spacetime):

\[
d\mu = d^4p = \frac{1}{4!} \epsilon^{\mu\nu\rho\sigma} dp_\mu dp_\nu dp_\rho dp_\sigma
\]

(5.2)

Given the fact that I am deforming the symmetry structure of spacetime (for example assuming the symmetry algebra to be $\kappa$-Poincaré instead of the classical Poincaré one), it is intuitive that I will need a deformed scalar product, in order for it to remain invariant under the symmetry action.

I will still assume the commutativity between momentum components, and so the possibility to use them as base coordinates for our Hilbert space. In general our integration measure in kinematical momentum space will have the form:

\[
d\mu(p) = \gamma(p) d^4p
\]

(5.3)

Where the function $\gamma$ is chosen so to ensure the deformed invariance of the measure. This integration measure deformation suggests us that we are considering momenta no more as a linear manifold, but as a curved space, in the spirit seen at the classical level in relative locality theories \footnote{These theories try to describe deformations of the behaviour of classical (not quantum) particles when very high energies are approached, using field theories on curved momentum space.}. In these theories we allow momentum manifold to have nontrivial properties, like curvature and a non linear addition. These theories try to describe deformations of the behaviour of classical (not quantum) particles when very high energies are approached, using field theories on curved momentum space.
We can interpret our deformation of quantum mechanics as a quantum version of the classical relative locality theories, allowing us to describe quantum particles having a curved momentum space. We cannot however describe interactions, having to postpone the task to the introduction of a quantum field theory on non-commutative spacetime.

As practical example I can show how the process of deformation of the scalar product work for the $\kappa$-Minkowski spacetime. I shall go through the process of representation of the symmetry group to derive the scalar product, but let’s assume for a moment the representation has already been detailed (I will go through the process in the next chapter). In 1+1 spacetime dimensions, the only Lorentz generator is the one of boost, and as we will see it has the form:

$$\eta = \left( \frac{e^{2\ell p_0} - 1}{2\ell} + \frac{\ell}{2} p_1^2 \right) q_1 - p_1 q_0$$  \hspace{1cm} (5.4)

The action on momentum components is:

$$p_0 \rightarrow p_0 + \xi p_1 \hspace{1cm} p_1 \rightarrow p_1 + \xi \left( \frac{e^{2\ell p_0} - 1}{2\ell} + \frac{\ell}{2} p_1^2 \right)$$  \hspace{1cm} (5.5)

Consequently differentials will transform as:

$$dp_0 \rightarrow dp_0 + \xi dp_1 \hspace{1cm} dp_1 \rightarrow dp_1 + \xi \left( e^{2\ell p_0} dp_0 + \ell p_1 dp_1 \right)$$  \hspace{1cm} (5.6)

From these transformation properties we can derive that the undeformed volume element is not invariant anymore under the deformed Lorentz boost:

$$d^2 p \rightarrow (1 + \xi \ell p_1) d^2 p$$  \hspace{1cm} (5.7)

With a similar calculation one can instead prove the invariance of the measure:

$$d\mu(p) = e^{-\ell p_0} d^2 p$$  \hspace{1cm} (5.8)

$$e^{-\ell p_0} \rightarrow (1 - \xi \ell p_1) e^{-\ell p_0} \Rightarrow e^{-\ell p_0} d^2 p \rightarrow e^{-\ell p_0} d^2 p$$  \hspace{1cm} (5.9)

This result implies that any hermitian operator has to commute with the function $e^{-\ell p_0}$, condition that can guide us in the choice of our new coordinates. As a side result of this choice we expect the commutation rules between them to be the ones of $\kappa$-Minkowski spacetime. We will see in the next chapter that our choice will be a little different than the one explained, because in the development of the theory the deformed scalar product was a byproduct of the deformation of coordinates, not the reverse. Of course the different choice does not alter the physical effects of the deformation.

As a side step we can note that this deformed measure can be interpreted (and indeed it should be interpreted) as an invariant integration measure on a curved manifold, for which we can derive the condition:

$$\sqrt{|g(p)|} = e^{-\ell p_0}$$  \hspace{1cm} (5.10)

in accordance to the condition to have an invariant integration measure on a manifold. This association give us an hint to a nice geometrical interpretation of
non-commutative spacetime theories: the deformation acts primarily on momentum space, having as a result the curvature of the momentum (previously linear) manifold. To have an invariant scalar product on the Hilbert space defined on such a manifold we have to adopt the invariant integration measure, which in turn forces us to consider non-trivial, non-commutative hermitian coordinate operators, which in turn comply with the defining commutation rules of the particular quantum spacetime. This geometrical characterization of non-commutative spacetime theories is subject to ongoing study.

5.2 Coordinate operators

I want now to study the effects of the deformation on the kinematical observables. In particular the idea is to find the operators, in the deformed kinematical space, corresponding to the spacetime coordinates \( x^\mu = -i \partial_{p^\mu} \) of the commutative space. There is a double reason to change the kinematical coordinates; apart from trying to replicate the non trivial commutation rules of non-commutative spacetime, I also note that the undeformed coordinates are no more hermitean operators in the deformed space as already noted. Being the integration measure non trivial, we cannot use standard integration by part to conjugate the derivative operators. From now on we will relabel undeformed derivative operators \( q^\mu \), and the deformed coordinates \( x^\mu \):

\[
 x^\mu = \varepsilon^\mu_\nu(p)q^\nu + \zeta^\mu(p) = -i\varepsilon^\mu_\nu(p)\frac{\partial}{\partial p^\nu} + \zeta^\mu(p) \quad (5.11)
\]

The functions \( \varepsilon^\mu_\nu \) and \( \zeta^\mu \) are not completely arbitrary, in that we ask the deformed coordinates to be hermitian operators. This translates to the conditions on the coefficients:

\[
 \varepsilon^\mu_\nu \in \mathbb{R}, \quad \partial^\mu \left( \gamma \varepsilon^\mu_\nu \right) + 2\gamma \text{Im} \left( \zeta^\mu \right) = 0 \quad (5.12)
\]

In addition we limit ourselves to affine functions of the undeformed coordinates, mainly to preserve the translation invariance of the theory: we don’t want commutators between coordinates and momenta to be coordinate dependent, otherwise we would have position-dependent translations. We are assuming our spacetime to be flat, so we reject this hypothesis.

These conditions are however not enough to determine the coefficient functions completely, so we have a certain ambiguity in the choice of kinematical coordinates.

As an example of ambiguity in the choice of kinematical coordinates we will consider two possibilities, again in \( \kappa \)-Minkowski spacetime. Starting from the deformation of the scalar product and embracing the geometric picture of the deformation, a natural choice of coordinates would be that of two killing vector fields on a manifold with metric \( g(p) \) such that:

\[
 \sqrt{|g(p)|} = e^{-\ell p_0} \quad (5.13)
\]

Two such vector fields with the right commutative limit are:

\[
 V^1 = \frac{\partial}{\partial p_1}, \quad V^0 = \frac{\partial}{\partial p_0} + \ell p_1 \frac{\partial}{\partial p_1} \quad (5.14)
\]
5.3 Translation parameters

In fact, selecting non-commutative coordinates of the form:

\[ x^1 = q^1 \quad x^0 = q^0 + \ell p_1 q^1 \]  (5.15)

we have both

\[ [x^1, x^0] = i\ell x^1 \]  (5.16)

and

\[ (x^1) = x^1 \quad (x^0) = x^0 \]  (5.17)

So coordinates in this form are a good example of representation on a quantum spacetime of the \( \kappa \)-Minkowski commutation relations.

As already stated, however, in practical cases it is easier to start from the commutation relations themselves, pick a reasonable function from them, and then check if the selected coordinates are good hermitian operators with respect to the deformed scalar product. All the results we derived in the next chapter were obtained with the different map:

\[ x^1 = e^{\ell p_0} q^1 \quad x^0 = q^0 \]  (5.18)

While it is evident that with such a definition the coordinates comply with (3.2), the \( x^0 \) is not hermitian. As explained in the next chapter, however, the non-hermitian part of this operator is not dangerous for our results. In addition such a representation makes the treatment of translation transformation a lot more intuitive, as we will see in the next section and chapter.

We stress once again that these different choices of kinematical coordinates cannot influence the physical results of the theory. The choice of kinematical coordinates are a gauge of the theory, so physical results cannot be dependent from it, although picking a particular set of kinematical coordinates can be useful in building up observables in the physical Hilbert space, as we will see in the next chapters. In principle one could start directly from the physical Hilbert space, and from covariance consideration for operators, and using the commutative limit condition, one could pick directly the physical position operators. This is however not a straightforward method, so it remains preferable to start from hermitian kinematical position operators, and than impose the hamiltonian constraint to obtain the physical coordinates.

In the subsequent chapters we will see the process applied more thoroughly to \( \kappa \)-Minkowski spacetime, of which we will be able to represent also the Hopf-algebra symmetry structure - another feature for which working in full phase space has an advantage. In other representations the symmetry generators are never operators acting on the same vector space as coordinate operators, making the treatment more cumbersome.

5.3 Translation parameters

In this section I will show in a clear way how the representation on the covariant kinematical Hilbert space is more useful than the usual ones, on which field theories are based. In the last part we saw how to implement in a satisfactory way translation symmetry in our spacetime, one condition was for parameters to satisfy the relations:

\[ [\epsilon^i, x^0] = i\epsilon^i \quad [\epsilon^\mu, \epsilon^\nu] = \epsilon^\rho \quad [\epsilon^\mu, x^j] = [\epsilon^0, x^\mu] = 0 \]  (5.19)
We saw, however, that in an irreducible representation we don’t have room for such parameters if we want them to be also coordinate independent.

In an irreducible representation the only spatial transformation parameters available are of the form:

\[ \epsilon^i = \epsilon_j x^j \]  

(5.20)

with \( \epsilon^{ij} \) an antisymmetric matrix. In other words, the are rotation parameters.

It is easy to think that to have translation parameters which are independent from coordinates but comply with non-trivial commutation rules with them, they have to be function on phase space, so they cannot be elements of a representation generated by coordinates alone.

The kinematical phase space in which we are representing non-commutative coordinates, on the other hand, give us an intuitive solution to the problem. In the second representation for coordinates, the one based on the commutation rules rather than the scalar product, the "non-commutative element" which introduces the non-triviality in the commutation rules is the exponential \( e^{i p_0} \), so we can consider non-commutative parameters of the form:

\[ \epsilon^i = e^{i p_0} a^i \quad \epsilon^0 = a^0 \quad a^\mu \in \mathbb{R} \]  

(5.21)

It is obvious that the parameters so defined will have the same commutation relations with the time coordinate as the space coordinates do. In addition, they do not depend on coordinates, so comply with all the requirements we needed in Noether analysis. A moment thought, in addition, let us conclude that translations made this way amount to standard translations in the auxiliary \( q^\mu \) coordinates, another point that makes things easier in our analysis.

To have energy dependent translations is a little baffling, like having energy dependent coordinates is. We have however another example of such behaviour (the mixing between position and momentum) in classical relative locality theories, of which we here propose a quantum generalization. As we will see in the next chapter, specifically for \( \kappa \)-Minkowski, this treatment gives a fully satisfactory, relativistic account of physical phenomena, the only concept that gets weakened is that of spacetime locality at large distances from the observer.

So we have now achieved a representation of translation parameters which satisfy us in terms of conserved charges. We would like to use this representation to perform now a field-theoretic Noether analysis (to prove conservation of symmetry generators in the single free particle setting is trivial), but the step of building a field theory out of this type of representation is still not completed, leaving the task to subsequent studies.

---

\(^2\)The variable \( p_0 \) is the one which generates translations in \( x^0 \), so it can rightfully be considered the energy of the particle.
Chapter 6

κ-Minkowski spacetime

I have introduced the κ-Minkowski spacetime in the previous chapters; it is one of the most studied examples of noncommutativity, and one of the cases with the most interesting aspects in our analysis. The commutativity relations between its coordinates, already introduced, are:

\[ [x^i, x^0] = i\ell x^i, \quad [x^i, x^j] = 0 \quad \forall i, j \]

another important feature of this spacetime in terms of representations is the set of translation parameters. It has been show that in terms of the non-commutative field theory the most convenient form of the parameters commutation form is:

\[ [\epsilon^i, x^0] = i\ell \epsilon^i, \quad [\epsilon^\mu, \epsilon^\nu] = [\epsilon^i, x^j] = [\epsilon^0, x^\mu] = 0 \quad \forall i, j \]

In the next section I will briefly review the results about coordinates and parameters representation already exposed in the last chapter. In the section I will go in detail in explaining the representation of the full symmetry algebra of κ-Poincaré on the same Hilbert space on which coordinate operators are defined. Such an achievement will ease the discussion on the symmetries of this spacetime. After a short description of the relative locality effects in this quantum theory I will go on to the description of the physical Hilbert space, in which we will see spacetime fuzziness affects propagation of particles wavepackets.

6.1 Coordinates representation

As shown in the last chapter we have in principle multiple choices on how to represent κ-Minkowski coordinates on the kinematical Hilbert space. In our work [33] we limited ourselves to the 1+1 dimensional case, to avoid inessential calculational problems, and picked the most intuitive choice for \( x^0 \) and \( x^1 \) as functions of \( q^0 \) and \( q^1 \):

\[ x^0 = q^0, \quad x^i = e^{\ell p_0} q^i \]

and as we saw the same can be done for the parameters:

\[ \epsilon^0 = a^0, \quad \epsilon^i = e^{\ell p_0} a^i, \quad a^\mu \in \mathbb{R} \]
6. \(\kappa\)-Minkowski spacetime

We see how spacetime coordinates become functions on phase space, or in other terms how particles momentum can alter its measured position in spacetime. Though we are still not in the physical space, so no phenomenological prediction can be drawn from the form of kinematically coordinates, we will see shortly how this features is replicated in the physical space too.

Another important concept related to coordinates to represent on the kinematical space is the boost parameter (it is the only Lorentz parameter, being us in the 1+1 dimensional case). Also for this kind of transformations we saw from the field theory that the Leibniz rule is an essential ingredient for Noether theorem; this brings us to the same representation for boost parameters that we had for spatial translations parameters:

\[ \xi = e^{f_{p_0}} b, \quad b \in \mathbb{R} \]  \hspace{1cm} (6.1)

6.2 Symmetry group representation

One of the major advancements this kind of representation allows us to make is the representation of the whole symmetry group of the \(\kappa\)-Minkowski spacetime on the same Hilbert space of which coordinates are defined. In this way we can easily see how symmetry transformation act on states, and in turn on coordinate operators.

We saw in the last part of the thesis which algebra (and coalgebra) symmetry generators have to reproduce in order to be a symmetry of \(\kappa\)-Minkowski. In particular we focus on translation generators, which have trivial algebra but not trivial coproduct:

\[ \Delta (P_0) = P_0 \otimes 1 + 1 \otimes P_0 \]  \hspace{1cm} (6.2)

\[ \Delta (P_i) = P_i \otimes 1 + e^{-f_{p_0}} P_i \]  \hspace{1cm} (6.3)

We know the Hopf-algebra generators to act on coordinates in the canonical way:

\[ P_\mu \triangleright x^\nu = i\delta_\mu^\nu \]  \hspace{1cm} (6.4)

To obtain this action we have to deform the canonical commutation relations on the Hilbert space:

\[ p_\mu \triangleright x^\nu \equiv [p_\mu, x^\nu] = \delta_\mu^\nu \delta_{1} e^{f_{p_0}} \]  \hspace{1cm} (6.5)

\[ p_\mu \triangleright x^\nu \equiv [p_\mu, x^\nu] = \delta_\mu^\nu \delta_{1} e^{f_{p_0}} \]  \hspace{1cm} (6.6)

\[ p_\mu \triangleright x^\nu \equiv [p_\mu, x^\nu] = \delta_\mu^\nu \delta_{1} e^{f_{p_0}} \]  \hspace{1cm} (6.7)

This definition of action on operators reproduces also the correct coproduct:

\footnote{We indicate Hopf-algebra symmetry generators in capital letters \(P_\mu\), standard Lie-algebra generators in lower case \(p_\mu\).}
\[
P_i \triangleright f(x)g(x) = e^{-\ell p_0}[p_i, f(x)]g(x) + e^{-\ell p_0} f(x)[g(x), p_i] + [p_i, f(x)]g(x) = (P_i \triangleright f(x)) g(x) + (P_i \triangleright g(x)) f(x) = P_i \triangleright f(x)g(x) \quad (6.8)
\]

Even if this thesis is more focused on translation properties of the theory, we shall note that we can correctly represent also boost generators, that in 1+1 dimensions have the algebra properties exposed in the second part, and a coalgebra that we report here for convenience:

\[
\Delta (N) = N \otimes 1 + e^{-\ell p_0} \otimes N \quad (6.10)
\]

It is straightforward to note that this coproduct rule is the same we found in the spatial translation case, so we can apply the same strategy we used in that case:

\[
N \triangleright \bullet = e^{-\ell p_0}[\eta, \bullet] \quad (6.11)
\]

where \(\eta\), to reproduce the right algebra, needs to be of the form:

\[
\eta = \left( \frac{e^{2\ell p_0} - 1}{2\ell} + \frac{\ell}{2} p_0^2 \right) q_1 - p_1 q_0 \quad (6.12)
\]

This is the representation of the full symmetry Hopf-algebra of our non-commutative spacetime, a result remarkable by itself, in that it allows us to study the action of a deformed symmetry on a quantum-mechanical Hilbert space in the same way as we do for a Lie-algebra symmetry.

For the translation sector, for example, we have that the infinitesimal action on an observable is:

\[
f(x + \epsilon) \approx (1 - i\epsilon P_0) f(x) \quad (6.13)
\]

where \(\epsilon^\mu\) here are the non-commutative translation parameters. Substituting the definition of the action of the Hopf-algebra generators we obtain:

\[
f(x + \epsilon) \approx f(x) - ia^\mu [p_\mu, f(x)] \quad (6.14)
\]

This of course is simply the adjoint action of the Lie-algebra action of the kinematical momentum generators, which translates on states to:

\[
\psi(q + a) \approx (1 + ia^\mu p_\mu) \psi(q) \quad (6.15)
\]

and in turn this, being the standard action, can be exponentiated to:

\[
\psi(q + a) = e^{ia^\mu p_\mu} \psi(q) \quad (6.16)
\]

We point out that states are defined as functions of the kinematical observables \(q\), not the non-commutative \(x\), as they obviously don’t constitute a complete set of commuting observables. Another interesting fact to note is that, with the parameter
chosen to have Leibniz rule, the deformed translations in $x$ are standard translations in the $q$ variables. This fact implies relative locality effects when we study the kinematical observables $x$.

We can carry on the same reasoning for the boost action (both the coproduct and the parameter are the same as in the spatial translation sector), so finding a boost infinitesimal action on observables:

$$f(x)' \approx (1 - i \xi N) \triangleright f(x) = f(x) - ib[\eta, f(x)]$$

(6.17)

We can again translate this infinitesimal action on states as:

$$\psi(q)' \approx (1 + i b \eta) \psi(q)$$

(6.18)

and exponentiate the action to:

$$\psi(q)' = e^{ib\eta} \psi(q)$$

(6.19)

From now on we are able to perform a full symmetry analysis of our spacetime, both in terms of states and in terms of observables. Moreover, this treatment is an infinitesimal deformation of the standard analysis we could perform in quantum mechanics, so we can apply a number of standard procedures that are not available in other types of representations.

Having derived the full symmetry structure of the kinematical Hilbert space, we can carry on and derive the invariant integration measure we need to define the scalar product. The trasformation properties of momentum components under boost are:

$$p_0' = p_0 - bp_1$$

(6.20)

$$p_1' = p_1 - b \left( e^{2i\ell p_0} - \frac{1}{2\ell} + \frac{\ell}{2} p_1^2 \right)$$

(6.21)

We proved that an invariant measure under this set of transformations is:

$$d\mu(p) = e^{-\ell p_0} dp_0 dp_1$$

(6.22)

The introduction of an invariant measure concludes the "ingredient list" to have a well defined quantum mechanics, at least in the kinematical space; we have states expressed as functions of canonical coordinates $\psi(q)$ or - more often and here the only case exploitable - as functions of momentum components $\psi(p)$. On these states we have a well defined scalar product, in the form

$$\langle \phi | \psi \rangle = \int \overline{\phi}(p) \psi(p) e^{-\ell p_0} d^2p$$

(6.23)

One example by which momentum space representation is more usable than coordinate space is that the deformed scalar product is straightforward to present in momentum space, not at all in coordinate space.
We can now go on and think on the effects of the scalar product deformation on hermiticity of operators: of course any function of momentum components will not change its hermiticity properties, and the same can be done for the boost generator, given the fact that the measure was built explicitly to be invariant under boost.

The only other operators we want to check are the spacetime coordinates $x^\mu$. While it is evident that the spatial coordinate $x^i = e^{\ell p_0} q^i$ remain hermitian, the same cannot be said for the time coordinate $x^0 = q^0$. For it we have the property:

$$\left( x^0 \right)^\dagger = x^0 + i \ell$$

(6.24)

While this is a little upsetting, it is not by itself an irreparable defect of the theory: first of all, as we said we have some degree of freedom in the definition of the kinematical coordinates, so we can redefine the new time variable as:

$$\tilde{x}^0 = x^0 + i \frac{\ell}{2}$$

(6.25)

thus regaining hermiticity (at the expenses of the transformation properties).

As a second point, we note that we can physically experience only time differences, so a constant non hermitian part in the time coordinate does not have measurable consequences.

The last reassuring remark - partially linked to the last two - is that we are still in the kinematical Hilbert space, that we stated has not physically measurable observables in $i\ell$.

6.3 Quantum relative locality

We have laid out the basic structure of the $\kappa$-Minkowski deformed Hilbert space. With the definition of the scalar product and the deformed (kinematical) coordinates we can evaluate the effects of the deformation of expectation value on localization of spacetime points.

In literature there are examples of momentum-dependent localization, but they are all classical examples, in which quantum and gravitational contribution are negligible. Here we recover the same kind of effects, but regarding a quantum particle. The first effect is on the expectation values of coordinate observables. We obtain, for the generic state labeled by four position parameters $q^\mu$ and factorizable in temporal and spatial part:

$$\langle \tilde{x}^0 \rangle = q^0$$

(6.26)

$$\langle x^1 \rangle = (e^{\ell p_0} q^1) = (e^{\ell p_0}) q^1$$

(6.27)

So we can see as a generic feature the dependence of positions from the energy of the state. In particular with $\ell > 0$ we see as farther states that have an higher energy mean value. We can restrict our analysis to a particular class of states, to

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3It is worth specifying anyway that we can find a set of hermitian kinematical coordinates with the right transformation properties, in the form of killing vectors on momentum space manifold.

4Here for definiteness we will use the observable $\tilde{x}^0$ as time coordinate.
show the precise features of this dependence. We take gaussians labeled by mean values and uncertainties:

\[ \psi_{\mu, \bar{\sigma}_{\mu}}(p) = N e^{-\frac{(p_0 - \bar{p}_0)^2}{4\bar{\sigma}_{0}^2}} e^{i(p_0 \bar{q}_0 - p_1 \bar{q}_1)} \]  

(6.28)

With \( N \) the normalization constant:

\[ N^2 = \frac{e^{\ell p_0} e^{-\frac{\ell^2 \bar{\sigma}_{0}^2}{2}}}{2\bar{\sigma}_{0}\bar{\sigma}_{1}} \]  

(6.29)

With an explicit form for the states we can also evaluate a novel effect that we cannot find in relative locality literature, given its quantum origin. We can study how uncertainties on these states, or the extension of the spacetime regions described by them, change with the distance from the observer. This is of course a direct effect of the deformation, given that in galilean quantum mechanics uncertainties on positions are translation-invariant quantities.

While for the time coordinate we have undeformed results:

\[ \langle \tilde{x}_0 \rangle = \bar{q}_0 \]  

(6.30)

\[ \delta \tilde{x}_0 = \sqrt{\langle \tilde{x}_0^2 \rangle - \bar{q}_0^2} = \frac{1}{2\bar{\sigma}_{0}} \]  

(6.31)

for the space coordinate expectation values we have:

\[ \langle x_1 \rangle = \langle q_1 \rangle \left( e^{\ell p_0} \right) = \bar{q}_1 e^{\ell p_0} e^{-\frac{\ell^2 \bar{\sigma}_{0}^2}{2}} \]  

(6.32)

\[ \delta x_1 = e^{\ell p_0} \left[ \frac{1}{4\bar{\sigma}_{1}^2} + \bar{q}_1^2 \left( 1 - e^{-\ell^2 \bar{\sigma}_{0}^2} \right) \right]^{1/2} \]  

(6.33)

From the uncertainties value it is evident the distinctive feature of a fuzzy spacetime: we cannot have a limit gaussian describing a sharply localized point. While in the commutative, undeformed case we could take the limit \( \sigma_{\mu} \to \infty \) to obtain \( \delta x_0 = \delta x_1 = 0 \), here taking \( \sigma_{0} \to \infty \) gives an irreducible contribution to \( \delta x_1 \).

We see however that this effect is dependent from the distance from the origin of the observer. The irreducible contribution to the spatial uncertainty is proportional to the space coordinate expectation value, so it can be made to vanish considering points localized in the reference frame origin.

This could lead to think (and many authors did [32]) that this kind of behaviour would spoil translation invariance, given the peculiar behaviour of the reference frame origin. We are however in a good position to show that there is no symmetry breaking in this formalism. In fact we have an explicit form of deformed translations which make this non-commutative spacetime perfectly relativistic.

It is easier to study translation covariance in the "Heisenberg picture", in which infinitesimal translations act on operators with the form:

\[ T \triangleright \bullet \approx (1 + i\epsilon \mu P_{\mu}) \triangleright \bullet = 1 \triangleright \bullet + ia^{\mu}[p_{\mu}, \bullet] \]  

(6.34)

when acting on coordinates, we obtain:
6.3 Quantum relative locality

\[ T \triangleright x_0 = x_0 - \epsilon_0 = q_0 - a_0 \quad (6.35) \]
\[ T \triangleright x_1 = x_1 - \epsilon_1 = e^{\ell p_0} (q_1 - a_1) \quad (6.36) \]

When applied to expectation values of coordinates and uncertainties, these transformations lead for the time coordinate to:

\[ \langle T \triangleright \tilde{x}_0 \rangle = q_0 - a_0 \quad (6.37) \]
\[ \delta \langle T \triangleright \tilde{x}_0 \rangle = \frac{1}{2\sigma_0} \quad (6.38) \]

while for spatial coordinates we find:

\[ \langle T \triangleright \tilde{x}_1 \rangle = (q_1 - a_1) e^{\ell p_0} e^{-\ell^2 \sigma_0^2} \quad (6.39) \]
\[ \delta \langle T \triangleright \tilde{x}_1 \rangle = e^{\ell p_0} \left[ \frac{1}{4\sigma^2} + (q_1 - a_1)^2 \left( 1 - e^{-\ell^2 \sigma_0^2} \right) \right]^{1/2} \quad (6.40) \]

These results for the translated expectation value clarify how the theory can be fully relativistic, though allowing for a greater localization for points near the origin of the reference frame. Each observer sees his neighborhood as "easier localizable", so for example can realize a sharp, pointlike localization of an event coinciding with his origin. The same event (i.e. the same state), observed from a translated observer will be characterized by the uncertainties \( \delta \langle T \triangleright x_\mu \rangle \) given by \( 6.39, 6.40 \). So the same spacetime region will be seen as more and more blurred the farther the second observer will be from the first.

This crucial point is shown in the next figure. There we see how two different observers translated with respect to each other perceive two spacetime regions localized both close and far from their origins. The two main features of this phenomenon are the already explained "relative fuzziness" of the points depending on their spatial distance from the origin, and the dependence of positions on the energy of the states (states with high energy can appear displaced with respect to the classical, low-energy ones).

Until now we described the features of \( \kappa \)-Minkowski kinematical Hilbert space, which at most can give a suggestion of what properties empty spacetime might show at very high energies. In particular the kinematical space can be used to set up the symmetry structure of the theory. To describe experimentally testable effects, however, we have to go on to description of particles moving in this spacetime. To do so we have to define the physical Hilbert space, and the first step of this procedure is the definition of the hamiltonian constraint of the theory. As already pointed out this constraint defines the dynamics of the theory, so it must be an invariant of the symmetry group. It is immediate to consider the mass casimir as constraint, in that it is invariant and reproduces the right dynamics in the commutative and galilean limit. The casimir of the symmetry algebra of our kinematical Hilbert space has the form \[ 33, 34, 35 \]:

\[ \Box = \left( \frac{2}{\ell} \right)^2 \sinh^2 \left( \frac{\ell p_0}{2} \right) - e^{-\ell p_0} p_1^2 \quad (6.41) \]
Figure 6.1. We illustrate the features of relative locality we uncovered for the $\kappa$-Minkowski quantum spacetime by considering the case of two distant observers, Alice and Bob, in relative rest (with synchronized clocks). In figure we have only two points in $\kappa$-Minkowski, each described by a gaussian state in our Hilbert space. One of the points is at Alice (centered in the spacetime origin of Alice’s coordinatization) while the other point is at Bob. The left panel reflects Alice’s description of the two points, which in particular attributes to the distant point at Bob larger fuzziness than Bob observes (right panel). And in Alice’s coordinatization the distant point is not exactly at Bob. Bob’s description (right panel) of the two points is specular, in the appropriately relativistic fashion, to the one of Alice. The magnitude of effects shown would require the distance $L$ to be much bigger than drawable. And for definiteness in figure we assumed $p_0 \simeq 2\sigma_0$ and $\sigma_1 \simeq \sigma_0$. 
so this is the function of momentum in the argument of the constraint delta function.
6.4 Physical Hilbert Space

As explained in the first part of the thesis we can go on to deform the scalar product of the theory to introduce the physical Hilbert space. The constraint is already found as the casimir of the $\kappa$-Minkowski symmetry group, but in addition we choose to consider only positive frequency states, so the new scalar product becomes:

$$\langle \phi | \psi \rangle_{\text{phys}} = \int dp_1 dp_0 e^{-\epsilon p_0} \delta(\Box) \Theta(p_0) \bar{\phi}(p) \psi(p) \tag{6.42}$$

with this deformed scalar product of course the normalization constant is changed accordingly; for the generic state $\psi$ we have:

$$N^2 = \left[ \int dp_1 dp_0 e^{-\epsilon p_0} \delta(\Box) \Theta(p_0) |\psi_{\eta_0,\sigma_0,\tilde{q}_0}(p)|^2 \right]^{-1} \tag{6.43}$$

Given the scalar product we could study localization of particles taking superposition with "detector" states [3]. We opt however for considering the effect of deformation on operators, studying localization from expectation values of coordinates and uncertainties. To do so we have to find the hermitian coordinate operators in the physical states, considering combinations of the kinematical of the kind already shown in the second chapter. In particular we will consider a deformed analogue of the Newton-Wigner operator:

$$A^1(T) = x^1 - \frac{\Box_{\ell, x^1}}{\Box_{\ell, x^0}} x^0 + \frac{\Box_{\ell, x^1}}{\Box_{\ell, x^0}} T - \frac{1}{2} x^0 \left[ \frac{\Box_{\ell, x^1}}{\Box_{\ell, x^0}} \right] \tag{6.44}$$

The deformation are the ones coming from the $\kappa$-Minkowski non-commutative coordinate representation, so we can see they are the physical remnants of the non-trivial (kinematical) commutation relations. Substituting the deformed coordinates we find:

$$A^1(T) = e^{\epsilon p_0} \left( q_1 - \mathcal{V} q_0 + \mathcal{V} T - \frac{1}{2} [q_0, \mathcal{V}] \right) \tag{6.45}$$

with the expression for $\mathcal{V}$ being:

$$\mathcal{V} = \left( \frac{e^{2\epsilon p_0} - 1}{2\epsilon p_1} \right) + \frac{\epsilon p_1}{2} \left[ \frac{\eta, p_1}{p_1, \eta} \right] \tag{6.46}$$

It can be noted that this family of operators, parametrized by the real number $T$, contains the same information as the single operator $A^1 = A^1(0)$, from which all the other values of $T$ can be recovered by a time translation. We can see the expectation value of this operator $A^1$ as the intercept of the particle worldline with the observer detectors when the coordinate time is zero. Moreover given the translational symmetry of the theory we can study position uncertainties at any mean value of $A^1$, and recover the information about any other state with a spatial translation.

To study particle localization we use the gaussian states introduced in 6.28 as typical, well localized states. In addition for simplicity we assume $\sigma_1 \ll \sigma_0, p_1$; this
Physical Hilbert space

assumption allows us a saddle point approximation in \( p_1 \). After the saddle point approximation we are left with a gaussian in \( p_1 \) with 'effective width' \( \sigma \):

\[
\sigma^{-2} = \sigma_1^{-2} + \nabla^2 \sigma_0^{-2}
\]

With this approximation and the condition \( \bar{p}^\mu = 0 \) we obtain, up to quadratic order in \( \ell \) (from now on we drop the suffix \( \text{phys} \) to denote the physical scalar product, in that for the rest of the chapter we will be interested only in the physical Hilbert space):

\[
\langle A \rangle = 0
\]

\[
\delta A^2 = \langle A^2 \rangle \approx (1 + 2(\ell p_0)^2) \frac{1}{4\sigma^2}
\]

The last results give us the picture for an observer which sees the particle crossing her spacetime origin. It is interesting to see what happens to the same state as observed by a system translated with respect to the first one, or in other words the state for an observer reached from the particle after a finite propagation time (it is useful to remember that in the physical space states describe worldlines of particles, so a single state describes the whole history of a particle). The family of translated observers whose origins are crossed by the particle worldline can be singled out by the condition:

\[
\langle \psi_{a_0,a_1}|A|\psi_{a_0,a_1} \rangle = 0
\]

where \( \psi_{a_0,a_1} \) are the translated states with parameters connecting the new observers with the old one. It is easy to argue - for example thinking to the classical limit - that the family of parameters solving this condition (or, alternatively, the family of observers 'on the worldline') is given by:

\[
a^1 = \nabla a^0
\]

It is interesting to evaluate the uncertainty of such coordinate for these operators; like one would expect from results on the kinematical space, we obtain

\[
\delta A^2 = \langle \psi_{a_0,a_1}|A^2|\psi_{a_0,a_1} \rangle \approx (1 + 2(\ell p_0)^2) \frac{1}{4\sigma^2} + \ell^2 \sigma^2 a_0^2 + \ell^2 \sigma^2 a_0^2
\]

We can interpret the first observer, the one with \( a_0 = a_1 = 0 \), as the observer at the source of the particle, and he has the minimal uncertainty - which could be reduced to zero - while the second one as the observer at the detector, which reveal the particle after some propagation distance. We see moreover that - like in the kinematical space - once we are outside the origin there is a lower bound on coordinate uncertainty.

In addition to position uncertainties we can also evaluate energy and momentum uncertainties; for example we find for the energy:

\footnote{Obviously we have also a contribution to the uncertainty given by the standard quantum mechanics wavepacket spreading, but here we are interested only in contributions from the scale \( \ell \).}
\[ \delta p_0^2 \approx (1 - 2/\ell p_0) \sigma^2 \quad (6.53) \]

This is a remarkable result because in precious heuristic arguments was always assumed the presence of an irreducible Planck-scale contribution to momentum uncertainties. Here instead \( \sigma \) is a free parameter, so uncertainties on momentum components can be made small at will, without constraints from Planck-scale contribution.
Chapter 7

Snyder spacetime

Snyder spacetime was the first example of non-commutative spacetime proposed in literature [5, 36, 37, 38]. It was proposed in the beginning to cure divergences in quantum field theory that were later solved with the renormalization theory; after the introduction of the idea of quantum-gravity effects on the small scale structure of spacetime this model of non-commutativity gained a new popularity, and became one of the most studied examples of quantum spacetime. The characteristic commutation relations between Snyder coordinates are:

\[
[x^\mu, x^\nu] = i\ell^2 M^{\mu\nu} = i\ell^2 (x^\mu p^\nu - x^\nu p^\mu)
\]  

(7.1)

\[M^{\mu\nu}\] being the generators of the Lorentz group. The form of this commutation relations suggest another fundamental feature of Snyder spacetime: its covariance under the standard - undeformed - Lorentz group. Indeed the main result of Snyder’s original paper was the discovery of a spacetime with - as we will see - a discrete structure, which anyway preserves the Lorentz symmetry structure of Minkowski spacetime. The analysis of such commutation relations will provide an interesting insight in the substantial difference between observables on the kinematical Hilbert space and the ones on the physical Hilbert space. Unlike the \(\kappa\)-Minkowski case, here the transition between kinematical and physical space hides unexpected results [39].

7.1 Coordinate representation and discreteness

Snyder in his paper proposed a representation of his non-commutative coordinate of the form:

\[x^\mu = (\delta^\mu_\nu - \ell^2 p^\mu p^\nu) q^\nu \equiv \Xi^\mu_\nu q^\nu\]  

(7.2)

where \(q^\mu\) are the standard derivative operators \((-i\partial/\partial p^\mu\)) on a - curved - momentum space. Evaluating the commutators we find indeed:

\[
[x^\mu, x^\nu] = \left(\Xi^\nu_\rho \Xi^\rho_\sigma q^\sigma - \Xi^\nu_\sigma \Xi^\sigma_\rho q^\rho\right) q^\rho = -i\ell^2 \left(\Xi^\nu_\rho p^\rho - \Xi^\rho_\nu p^\nu\right) q^\rho = i\ell^2 \left(x^\mu p^\nu - x^\nu p^\mu\right)
\]  

(7.3)
where we used the symmetry of the matrix $\Xi^{\mu\nu}$.

One could note that the coordinate operators $x^\mu$ are not hermitian with respect to the standard kinematical scalar product integration measure $d^4p$. Snyder proposed this was because momentum space was not the standard flat Minkowski space, but a De Sitter space of which spacetime coordinates were generalized 'rotation generators'.

The correct volume element on such a momentum space, which makes the $x^\mu$ operators hermitian, is:

$$d\mu(p) = \frac{d^n p}{(1 - \ell^2 p^2)^{\frac{n+1}{2}}}$$

where $n$ is the number of spacetime dimensions.

So the complete expression for the scalar product in this kinematical Hilbert space is:

$$\langle \phi | \psi \rangle = \int \frac{d^n p}{(1 - \ell^2 p^2)^{\frac{n+1}{2}}} \bar{\phi}(p) \psi(p)$$

and with respect to this scalar product the position operators $x^\mu$ are indeed hermitian.

We note again that in no step of the construction of this kinematical space we introduced Lorentz symmetry-breaking elements: both commutation relations and the definition of spacetime coordinates contain no privileged vectors, and the deformed scalar product is defined in terms of Lorentz scalars. We can conclude that the standard Lorentz group is still a good symmetry group for this spacetime, as Snyder claimed in his paper.

A somewhat surprising result, given the Lorentz covariance, is that this spacetime is also discrete, in that its cartesian space coordinates form a quantum lattice of spacing $\ell$. To prove this result we will stick to 4 spacetime dimensions, and we will define an $SO(4)$ algebra starting from space coordinates and rotation generators:

$$L_{ij} \equiv M_{ij}, \quad L_{i4} \equiv \ell^{-1} x_i, \quad L_{AB} = -L_{BA}$$

Given the previously defined commutation relations we have:

$$[L_{AB}, L_{CD}] = i(\delta_{AC}L_{BD} - \delta_{BC}L_{AD} - \delta_{AD}L_{BC} + \delta_{BD}L_{AC})$$

These relations can be made simpler with the change of basis:

$$K_i^\pm = \frac{1}{2} \left( L_i \pm \ell^{-1} x_i \right)$$

with $L_i = \epsilon_{ijk}M^{jk}$.

With the new basis the algebra of these operators becomes:

$$[K_i^+, K_j^+] = i\epsilon_{ijk}K_k^\mp$$

$$[K_i^+, K_j^-] = 0 \quad \forall i, j$$

$$K_i^+ K_i^+ = K_i^- K_i^-$$
The relevant $SO(4)$ algebra decouples in the product of two $SU(2)$ subalgebras, with a common casimir. From this we can derive that any representation we can find of the space coordinates and the rotation generators must also be a representation of two copies of the angular momentum algebra, with the same total angular momentum. The states in this representation can thus be labeled by the eigenvalues of $K_3^+$, $K_3^-$, and the casimir $K_3^+K_3^-$:

\[ K_3^+ |j, m_+, m_-\rangle = m_+ |j, m_+, m_-\rangle \]  
\[ K_3^- |j, m_+, m_-\rangle = m_- |j, m_+, m_-\rangle \]  
\[ K_3^+ K_3^+ |j, m_+, m_-\rangle = j(j+1) |j, m_+, m_-\rangle \]

In any of these representations we can easily recover the spectrum of the spatial coordinates; for example we have for $x_3$:

\[ x_3 = \ell \left( K_3^+ - K_3^- \right) \]  
\[ x_3 |j, m_+, m_-\rangle = \ell (m_+ - m_-) |j, m_+, m_-\rangle \]

Given the discrete nature of the eigenvalues $m_\pm$ we derive that the spectrum of the $x_3$ operator is a one-dimensional lattice of spacing $\ell$, and infinite degeneracy of each eigenvalue. Of course the same reasoning could be applied to the coordinates $x_1$ or $x_2$, but only one of each at a time, given the commutation relations between them. We can conclude that the spatial sector of Snyder spacetime is a quantum cubical lattice of lattice spacing $\ell$, even if the Lorentz covariance of the theory is preserved. To show an example of such a spacetime was the main achievement of Snyder’s original paper.

### 7.2 Physical Hilbert space

We have to remember that all the results obtained in the last section - and which are possible to find in literature from Snyder’s original paper on - are valid on a kinematical level. We still haven’t introduced any particle. To obtain something of actually, physically measurable we have to impose the hamiltonian constraint and find observables on the physical Hilbert space.

Here the structure of the symmetry group becomes of primary importance. As already said many times, the hamiltonian constraint has to be an invariant element of the symmetry group, i.e. the casimir of the symmetry algebra. We have however seen how the Snyder spacetime symmetry group is the undeformed Lorentz group (in Snyder original paper there are some issues raised about translations, but the Lorentz casimir is of course independent from translations).

The constraint will be, for this reason, the same of the undeformed Minkowski spacetime:\footnote{One could argue that the constraint should be some non trivial function of the undeformed $\Box$, but when inserted in the delta function and considering the right classical limit, the result would be the same.}

\[ \Box_\ell = \Box = p^2 - m^2 \]  

\[ \Box_\ell = \Box = p^2 - m^2 \]
Considering also the deformed kinematical integration measure, the physical scalar product becomes, in four spacetime dimensions:

\[
\langle \phi | \psi \rangle_{phys} = \int \frac{d^4p}{(1 - \ell^2 p^2)^2} \frac{\delta \left(p^2 - m^2\right)}{\phi(p) \psi(p)}
\]

This is the first hint to a trivialization of Snyder spacetime on the physical space: the deformation of the scalar product amount, considering the constraint to which particle are subjected, just to a multiplication by a constant - and this can be reabsorbed in the normalization of states. So on the physical sector the Snyder scalar product becomes completely trivial.\(^2\)

As next step we should now consider the physical analogue of the position operators \(x^\mu\), to see if there is a trace of the discretization found in the kinematical space. The operator we will consider as coordinates of interest are of two different classes: the generalization of the Newton-Wigner relativistic position operators

\[
\chi^\mu_p(T) = x^\mu - \frac{p^\mu}{v \cdot p} \cdot x + \frac{p^\mu}{v \cdot p} T + h.c.
\]

from which the Newton-Wigner operators can be regained with the choice \(v^\mu = \delta^\mu_0\), and the covariant position operators \([17]\):

\[
\chi^{\mu}_p(T) = x^\mu - \frac{p^\mu}{p^2} \cdot x + \frac{p^\mu}{p^2} T + h.c.
\]

Here the kinematical coordinates \(x^\mu\) are of course the \(\ell\)-deformed, non-commutative coordinates introduced by Snyder, given we want to study the effects of their non-commutativity on the physical observables. For convenience we report here their expression in term of undeformed kinematical coordinates \(q^\mu\)\(^7,2\):

\[
x^\mu = \left(\delta^\mu_0 - \ell^2 p^\mu p_0\right) q^\nu
\]

Substituting these expressions in the physical coordinates (here we indicate with an index \(\ell\) the deformed operators to make clear the dependence from the deformation parameter) we obtain:

\[
\chi^\mu_{\ell}(0) = x^\mu - \frac{p^\mu}{v \cdot p} \cdot x + h.c. = \\
= q^\mu - \ell^2 p^\mu v \cdot q - \frac{p^\mu}{v \cdot p} v_\nu \left(q^\nu - \ell^2 p^\nu v \cdot q\right) + h.c. = \\
= q^\mu - \frac{p^\mu}{v \cdot p} \cdot q + h.c. = \chi^\mu_{\ell=0}(0)
\]

\(^2\)There could be a remnant of the \(\ell\) deformation in the fact that for \(p^2 = \ell^{-2}\) we would have a singular integration measure, but it is not a very interesting feature, at this stage.
So the physical observables are completely independent from the deformation parameter $\ell$, as we would have expected given the undeformed Hilbert structure of the space on which they are defined. Of course the calculation is not altered in any aspect considering the coordinates $\chi^\mu_p$ or the general $T \neq 0$ case.

So, even if deformation parameter $\ell$ is of course implied in the definition of the kinematical - and consequently of the physical coordinates - once on the physical space all coordinate operators considered are $\ell$-independent, and as a consequence they retain the same commutation relations they had in the standard Minkowski spacetime. This gives us the second hint that the physical sector of Snyder spacetime could be trivial, but for now it could be a particularly unlucky coincidence that made us pick two classes of operators which trivializes once the constraint is imposed; in principle there could exist another coordinate operator whose dependence from $\ell$ is unaltered after the imposition of the constraint. We will show in the next section that this is not the case, and Snyder spacetime in this setting loses all the non-triviality given in the kinematical space by the dependence from $\ell$.

### 7.3 General deformed observables

The examples of observables which become trivial when considered on the physical Hilbert space are already pretty meaningful, in that they entail all the relativistic coordinate operators previously considered in literature to the knowledge of the author. We are however able to prove an even more general result; we can consider a general quantum observable on the physical Hilbert space of Snyder spacetime, and show that it cannot depend on the deformation parameter $\ell$. In the proof we focus on quantum observables which admit a smooth classical limit, and show that this classical limit cannot depend on $\ell$, so neither could do its quantum version. In order to commute with the hamiltonian constraint such an observable $f(p,q)$ must be such that:

\[
[\Box, f] \propto p^\mu \frac{\partial f}{\partial q^\mu} = 0 \tag{7.22}
\]

So if $f(p,q)$ has to be an observable on the physical space, the vector $\frac{\partial f}{\partial q^\mu}$ has to be orthogonal to the momentum $p^\mu$. It is possible to find a basis for the space orthogonal to the momentum in terms of the boost generators in phase space:

\[
\eta^i = p^i q^0 - p^0 q^i \tag{7.23}
\]

Considering the three vectors

\[
\eta^{i,\mu} = \frac{\partial \eta^i}{\partial q^\mu} \tag{7.24}
\]

it is easy to check that they are independent and that $p_\mu \eta^{i,\mu} = 0$. We can conclude from this that the vector function $\frac{\partial f}{\partial p^\mu}$ admits an expression in the form:

---

3Even if the definition of the physical Planck length $L_P$ is based on $\hbar$, in principle a deformation parameter with the dimension of a length is independent from the quantum or classical nature of the system.

4In this section the symbol $[\cdot, \cdot]$ stands for Poisson brackets.
\frac{\partial f}{\partial q^\mu} = f'_i(p,q)\eta^i_{\mu} \quad (7.25)

Taking another derivative and imposing the symmetry between partial derivatives we have:

\frac{\partial^2 f}{\partial q^\mu \partial q^\rho}(p,q) = \frac{\partial f'_i(p,q)}{\partial q^\rho} \eta^i_{\mu} = \frac{\partial f'_i(p,q)}{\partial q^\mu} \eta^i_{\rho} \quad (7.26)

and from this it is easy to establish that:

\frac{\partial f'_i(p,q)}{\partial q^\rho} p^\rho = 0 \quad (7.27)

Therefore, just like \frac{\partial f}{\partial q^\mu} = f'_i(p,q)\eta^i_{\mu}, on has:

\frac{\partial f'_i(p,q)}{\partial q^\rho} = f''_{ij}(p,q)\eta^j_{\rho} \quad (7.28)

The argument can be easily iterated to the expression:

\frac{\partial^n f(p,q)}{\partial q^\mu \partial q^\nu \ldots \partial q^\nu} = f^{(n)}_{ij \ldots k}(p,q)\eta^i_{\mu}\eta^j_{\rho} \ldots \eta^k_{\nu} \quad (7.29)

This constraint implies that \( f(p,q) \) can be a function of the coordinates \( q^\mu \) only through the \( \eta^i \), such that

\( f(p,q) = \tilde{f}(p,\eta) \quad (7.30) \)

We can now consider the possibility of \( \ell \)-deforming the general physical observable \( \tilde{f}(p,\eta) \) we found. At first order in the deformation parameter the new observable has to be of the form:

\( \tilde{f}_\ell(p,\eta) = \tilde{f}(p,\eta) + \frac{\partial \tilde{f}(p,\eta)}{\partial \eta^i} \delta_{\ell} \eta^i + \frac{\partial \tilde{f}(p,\eta)}{\partial p^\mu} \delta_{\ell} p^\mu \quad (7.31) \)

where \( \delta_{\ell} \eta^i \) and \( \delta_{\ell} p^\mu \) are the \( \ell \) dependent charges in \( \eta^i \) and \( p^\mu \) due to the deformation. For the case we are treating here however, the Snyder deformation, we have \( \delta_{\ell} p = 0 \) (in this representation \( p^\mu \) are just the components of the kinematical momentum canonical conjugate to \( q^\mu \)). Also the generators \( \eta^i \) are unchanged with respect to the classical case, given the Snyder spacetime has the same Lorentz symmetry structure as the Minkowski one.

Therefore here, in the Snyder spacetime case, all of the physical observables are undeformed with respect to the commutative case:

\( \tilde{f}_\ell(p,\eta) = \tilde{f}(p,\eta) \quad (7.32) \)

There is an apparent bug in this proof: one could think that the presence of an invariant scale \( \ell \) in the theory could allow us to introduce a new class of observables, of the kind

\( \tilde{f}_{[\ell]}(p,\eta) = \gamma(\ell p)\tilde{f}(p,\eta) \quad (7.33) \)
This is however not a limitation of our proof, for two reasons: the first is that this kind of observables exists also before deforming the theory, with the deformation scale replaced by any of the mass scales of the undeformed theory. For example one could consider the inverse electron mass to take the role of the scale $\ell$. The second, more important reason is that if one wants to retain the transformation properties of $\hat{f}(p, \eta)$ the deforming factor $\gamma(\ell p)$ has to be a scalar, so that for the single particle Hilbert space (unless some preferred external momentum is introduced) $\gamma$ could only be a function of the particle mass:

$$\gamma(\ell p)|_{\text{phys}} = \gamma(\ell m)$$  \hspace{1cm} (7.34)

and this quantity would be a numerical constant on the physical Hilbert space.

With this result we proved that - at least at the single particle, free theory level - is as trivial as the well known commutative Minkowski spacetime.
Conclusions

In this thesis I analyzed the problem of having a relativistic symmetry for a quantum, non-commutative spacetime. In particular I applied a covariant formulation of quantum mechanics to argue that Hilbert spaces of states of the same form of the quantum mechanical ones are the best environment to represent non-trivial commutation rules between coordinates.

After an introduction of the necessary mathematical elements of the theory I gave a presentation of the known theory of fields on noncommutative spacetime, plus some original results in this framework. The orientation of the exposure was on the symmetry algebra side, in that I wanted to give a unified representation of both coordinates and symmetry transformations. I used field theoretic Noether analysis to show that the transformation parameters, to allow for conserved charges, must have non-trivial commutation relations with coordinates, but have to commute with translation generators. I proved that no such parameters are admissible in a configuration space only representation, pointing out the need for a more general representation space than the ones based on field theory ideas.

The last part of the thesis was devoted to the application of the covariant quantum mechanics formalism to non-commutative spacetime representation, in the two particular cases of $\kappa$-Minkowski and Snyder spacetime, two of the most studied examples of quantum spacetime.

Two main results are common in these cases: I managed to represent on the same Hilbert space both coordinate operators, symmetry generators and symmetry parameters. In this way all the geometric information about the system is codified in a single state in the Hilbert space, and symmetry transformations are completely arbitrary; a situation completely similar to that of the standard, galilean quantum mechanics. As a consequence of the complete symmetry characterization of the two spacetimes, I proved that they are perfectly relativistic, in the sense that no preferred reference frame exists in neither of the two spaces. Such a complete and coherent description of these spacetimes and their symmetries was never achieved in literature, to the best knowledge of the author.

In $\kappa$-Minkowski spacetime I confirmed some previously expected results (deformed dispersion relations, energy-dependent velocity even for massless particles, uncertainty relations between space and time coordinates for events outside the observer’s origin). I generalized to the quantum world the relative locality concept, i.e. the dependence of locality of events from the observer. In this quantum generalization I saw that both the position and the degree of localization of a particle depend on the observer, in particular from his distance, and the energy he assigns to the wavepacket he is trying to localize. All these phenomena are, again, completely relativistic, and
were not present in literature before our work. As a completely new phenomenon I
derived a contribution to wavepacket spreading peculiar of spacetime-related effects.
I am able to replicate the standard wave-packet spreading of ordinary (relativistic)
quantum mechanics, and a quantum-gravity contribution to the spreading was found
and quantified.

For Snyder spacetime, on the other hand, the situation is completely different.
This space - the first introduced in literature - was always considered to have a
lattice structure in the three space directions. Apart from this assumption no one
was ever able to treat the dynamics of the quantum Snyder spacetime, the usual way
to proceed was to make heuristic hypothesis and to study the classical limit. When
applied to Snyder spacetime the new formalism proved perfectly fit; I could perform
the same kind of analysis that worked so well for quantum mechanics on Minkowski
and $\kappa$-Minkowski spacetime. In the case of Snyder spacetime when one considers
the distinction between kinematical and physical Hilbert spaces, it is easy to realize
that all the non-trivialities, included the quantum lattice structure of the space
coordinates, are limited to the kinematical space. Once the constraint is applied
to obtain the physical states, all the contributions of the non-trivial commutation
relations vanish, and the physical space is exactly the same as the undeformed
Minkowski one.

This wealth of new results and ease of interpretation in this covariant quantum
mechanical setting should convince us of the advantage of using full phase space
representations instead of coordinate-only ones. The next, interesting step to perform
would be to define a field theory based on the new kind of variables, in order to use
all the power of the symmetry structure representation, and complete the task of
performing a full fledged Noether analysis also in the contest of a many particle,
interacting theory.
Appendix A

Commutation relations between plane waves

From commutation relations (3.2) we can recover relations between arbitrary powers of coordinates:

\[ \hat{x}^0 \hat{x}^i = \hat{x}^i (\hat{x}^0 + i\lambda) \quad (A.1) \]
\[ (\hat{x}^0)^2 \hat{x}^i = \hat{x}^0 \hat{x}^i (\hat{x}^0 + i\lambda) = \hat{x}^i (\hat{x}^0 + i\lambda)^2 \quad (A.2) \]

It’s easy to see that, iterating this relations, we obtain for an arbitrary power:

\[ (\hat{x}^0)^n \hat{x}^i = \hat{x}^i (\hat{x}^0 + i\lambda)^n \quad (A.3) \]

Exchanging the role of spatial and temporal coordinates we obtain:

\[ \hat{x}^0 \hat{x}^i = \hat{x}^i (\hat{x}^0 + i\lambda) \quad (A.4) \]
\[ \hat{x}^0 (\hat{x}^i)^2 = \hat{x}^i (\hat{x}^0 + i\lambda) \hat{x}^i = (\hat{x}^i)^2 (\hat{x}^0 + i2\lambda) \quad (A.5) \]

and again, iterating:

\[ \hat{x}^0 (\hat{x}^i)^n = (\hat{x}^i)^n (\hat{x}^0 + in\lambda) \quad (A.6) \]

Now, generalizing these relations we have:

\[ (i\vec{k} \cdot \vec{x})^n (-ik_0 \hat{x}^0)^m = (i\vec{k} \cdot \vec{x})^{n-1} (-ik_0 (\hat{x}^0 - i\lambda))^m (i\vec{k} \cdot \vec{x}) \]
\[ (i\vec{k} \cdot \vec{x})^{n-1} (-ik_0 (\hat{x}^0 - i\lambda))^m (i\vec{k} \cdot \vec{x}) = (i\vec{k} \cdot \vec{x})^{n-2} (-ik_0 (\hat{x}^0 - i2\lambda))^m (i\vec{k} \cdot \vec{x})^2 \]
\[ \vdots \]
\[ (i\vec{k} \cdot \vec{x})^n (-ik_0 \hat{x}^0)^m = (-ik_0 (\hat{x}^0 - in\lambda))^m (i\vec{k} \cdot \vec{x})^n \quad (A.7) \]

In commuting a spatial and a temporal exponential we have to compute this type of commutators:
\[ e^{\imath \vec{k} \cdot \hat{x}} e^{-\imath k_0 \hat{x}^0} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \imath \vec{k} \cdot \hat{x} \right)^n (-\imath k_0 \hat{x}^0)^m \]

\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-\imath k_0 (\hat{x}^0 - \imath n \lambda))^m (\imath \vec{k} \cdot \hat{x})^n \]

\[ = \sum_{n=0}^{\infty} e^{-\imath k_0 \hat{x}^0 - n \lambda k_0} (\imath \vec{k} \cdot \hat{x})^n \]

\[ = e^{-\imath k_0 \hat{x}^0} e^{\imath \lambda \lambda_0 \imath \vec{k} \cdot \hat{x}} \quad (A.8) \]

Obviously to compute the commutator in the other direction it’s enough to change the sign to the exponential in the last exponent:

\[ e^{-\imath k_0 \hat{x}^0} e^{\imath \vec{k} \cdot \hat{x}} = e^{\imath \lambda \lambda_0 \imath \vec{k} \cdot \hat{x}} e^{-\imath k_0 \hat{x}^0} \quad (A.9) \]

From these relations we can also derive the commutators between single spatial coordinates and plane waves; they simply commute with spatial plane waves, and due the fact that in the last equations \( \vec{k} \) is an arbitrary numerical vector, we have:

\[ \hat{x}^i e^{-\imath k_0 \hat{x}^0} = e^{-\imath k_0 \hat{x}^0} e^{-\lambda \vec{k} \cdot \vec{x}^i} \quad (A.10) \]

The relations derived above for plane waves and single coordinates can be extended to arbitrary fields; for a single coordinate, by linearity of Fourier trasform, we have:

\[ \hat{x}^i f(\hat{x}) = \int d^4 k \tilde{f}(k) \hat{k}^i \hat{x}^j e^{-\imath k_0 \hat{x}^0} \]

\[ = \int d^4 k \tilde{f}(k) \hat{k}^i \hat{x}^j e^{-\imath k_0 \hat{x}^0} e^{-\lambda \vec{k} \cdot \vec{x}^i} \]

\[ = e^{-\lambda P_0} f(\hat{x}) \quad (A.11) \]

Similarly, for time coordinate we obtain:

\[ \hat{x}^0 f(\hat{x}) = f(\hat{x}) \hat{x}^0 - \lambda \vec{x} \cdot \vec{P} f(\hat{x}) \quad (A.12) \]

We don’t report the corresponding formulas for other Weyl maps, being these the only two used in the thesis.
Appendix B

Properties of deformed momenta

In the definition of the $k$-Poincaré casimir made their appearance the “deformed momenta” operators: $\tilde{P}_\mu$. Here we want to derive their algebraic and coalgebraic properties, heavily used in the derivation of a Noether’s theorem.

We recall the definition of $\tilde{P}_\mu$:

$$
\tilde{P}_0 = \frac{2}{\lambda} \sinh \left( \frac{\lambda P_0}{2} \right) = \frac{e^{\frac{\lambda P_0}{2}} - e^{-\frac{\lambda P_0}{2}}}{\lambda} \tag{B.1}
$$

$$
\tilde{P}_i = e^{\frac{\lambda P_0}{2}} P_i \tag{B.2}
$$

So we see that the properties of $e^{\frac{\lambda P_0}{2}}$ are of fundamental importance; for the coproduct we have:

$$
\Delta(e^{\alpha P_0}) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \Delta(P_0^n)
$$

$$
= \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \Delta(P_0)^n
$$

$$
= \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} (P_0 \otimes 1 + 1 \otimes P_0)
$$

$$
= e^{\alpha (P_0 \otimes 1 + 1 \otimes P_0)} = e^{\alpha P_0} \otimes e^{\alpha P_0} \tag{B.3}
$$

where we used the fact that $P_0$ has trivial coalgebraic properties.

Now we can turn on the main relations:
\[ \Delta(\tilde{P}_0) = \frac{1}{\lambda} \Delta(e^{\frac{\lambda P_0}{2}}) - \Delta(e^{-\frac{\lambda P_0}{2}}) = \frac{1}{\lambda} \left( e^{\frac{\lambda P_0}{2}} \otimes e^{\frac{\lambda P_0}{2}} - e^{-\frac{\lambda P_0}{2}} \otimes e^{-\frac{\lambda P_0}{2}} \right) \]

\[ = \frac{1}{\lambda} \left[ \left( e^{\frac{\lambda P_0}{2}} - e^{-\frac{\lambda P_0}{2}} \right) \otimes e^{\frac{\lambda P_0}{2}} + e^{-\frac{\lambda P_0}{2}} \otimes \left( e^{\frac{\lambda P_0}{2}} - e^{-\frac{\lambda P_0}{2}} \right) \right] \]

\[ = \tilde{P}_0 \otimes e^{\frac{\lambda P_0}{2}} + e^{-\frac{\lambda P_0}{2}} \otimes \tilde{P}_0 \quad (B.4) \]

\[
\Delta(\tilde{P}_i) = \frac{\lambda}{2} \Delta(e^{\frac{\lambda P_0}{2}}) \Delta(P_i) = e^{\frac{\lambda P_0}{2}} \otimes e^{\frac{\lambda P_0}{2}} \left( P_i \otimes 1 + e^{-\lambda P_0} \otimes P_i \right) = \tilde{P}_i \otimes e^{\frac{\lambda P_0}{2}} + e^{-\frac{\lambda P_0}{2}} \otimes \tilde{P}_i \quad (B.5) \]

Evidently these two relations can be combined in a single one:

\[
\Delta(\tilde{P}_\mu) = \tilde{P}_\mu \otimes e^{\frac{\lambda P_0}{2}} + e^{-\frac{\lambda P_0}{2}} \otimes \tilde{P}_\mu \quad (B.6) \]

Now we can turn to the commutation relations; an useful fact is that the commutator is a derivation, i.e. it respects Leibniz rule:

\[ [A, BC] = [A, B]C + B[A, C] \quad (B.7) \]

this enables us to conclude that for the exponentials (if the commutator in the r.h.s. itself commutes with the exponential) we have:

\[ [A, e^{\alpha P_0}] = \alpha e^{\alpha P_0} [A, P_0] \quad (B.8) \]

The condition in parenthesis is valid for all the generators of k-Poincaré, so we can write, for translations and rotations:

\[ [P_\mu, e^{\alpha P_0}] = \alpha e^{\alpha P_0} [P_\mu, P_0] = 0 \quad (B.9) \]

\[ [R_i, e^{\alpha P_0}] = \alpha e^{\alpha P_0} [R_i, P_0] = 0 \quad (B.10) \]

For boosts instead we have:

\[ [N_i, e^{\alpha P_0}] = \alpha e^{\alpha P_0} [N_i, P_0] = i\alpha e^{\alpha P_0} P_i \quad (B.11) \]

Again using properties of the exponentials we have, for \( \tilde{P}_0 \):

\[ [P_\mu, \tilde{P}_\mu] = 0 \quad (B.12) \]

\[ [R_i, \tilde{P}_\mu] = i\delta^k_\mu \epsilon_{ikj} \tilde{P}_j \quad (B.13) \]
\[ [N_i, \tilde{P}_j] = \frac{1}{\lambda} \left( [N_i, e^{\frac{\lambda P_0}{2}}] - [N_i, e^{-\frac{\lambda P_0}{2}}] \right) \]
\[ = i \cosh\left(\frac{\lambda P_0}{2}\right)P_i \]  
(B.14)

The derivation of commutation relations between boosts and spatial coordinates is a little more involved, and we report here only its result:

\[ [N_i, \tilde{P}_j] = i \delta_{ij} e^{\frac{\lambda P_0}{2}} \left( \frac{1 - e^{-2\lambda P_0}}{2\lambda} + \frac{\lambda}{2} |\vec{P}|^2 \right) - i\lambda P_i P_j \]  
(B.15)
Appendix C

Parameters commutation relations

In the thesis we repeatedly used the connection between spatial parameters and single coordinates:

\[ [\hat{x}^0, \epsilon^i] = i\lambda\epsilon^i \]  \hspace{1cm} (C.1)

and those required between spatial parameters and generic functions to ensure Leibniz rule for translations:

\[ \epsilon^i f(\hat{x}) = \left( e^{-\lambda P_0} f(\hat{x}) \right) \epsilon^i \]  \hspace{1cm} (C.2)

Here we want to show that the former condition implies the latter (the reverse implication is trivial):

\[ \hat{x}_0 \epsilon^i = \epsilon^i(\hat{x}_0 + i\lambda) \]  \hspace{1cm} (C.3)
\[ \hat{x}_0^2 \epsilon^i = \epsilon^i(\hat{x}_0 + i\lambda)^2 \]  \hspace{1cm} (C.4)
\[ \vdots \]
\[ \hat{x}_0^n \epsilon^i = \epsilon^i(\hat{x}_0 + i\lambda)^n \]  \hspace{1cm} (C.5)

so for an exponential we have:

\[
e^{-ik_0\hat{x}_0} \epsilon^i = \sum_{n=0}^{\infty} \frac{(-ik_0)^n}{n!} \hat{x}_0^n \epsilon^i = \sum_{n=0}^{\infty} \frac{(-ik_0)^n}{n!} \epsilon^i(\hat{x}_0 + i\lambda)^n = \epsilon^i e^{-ik_0\hat{x}_0} e^{\lambda k_0} \]  \hspace{1cm} (C.6)

This relation, exploiting again Fourier transform, becomes the one we were looking for:
\[
\begin{align*}
    f(\hat{\mathbf{x}}) \epsilon^i &= \int d^4 k \tilde{f}(k) e^{ik \cdot \hat{\mathbf{x}}} e^{-ik_0 \hat{x}_0} \epsilon^i \\
    &= \int d^4 k \tilde{f}(k) e^{ik \cdot \hat{\mathbf{x}}} e^{-ik_0 \hat{x}_0} e^{\lambda k_0} \\
    &= \epsilon^i e^{\lambda P_0} f(\hat{\mathbf{x}})
\end{align*}
\]

that is obviously equivalent to (C.2).
Appendix D

Integrals and divergences

A useful fact that we used a lot of times is the connection between null integrals and divergences; this calculations was inspired by the intuition that “all functions with null spacetime integrals are four-divergences”. The result of the calculation is that, tough not all functions respect this rule, indeed most of all do; more precisely, we assume a field \( \phi(x) \) to be rapidly decreasing to infinity, and so to have an infinitely many times differentiable Fourier transform \( \tilde{\phi}(k) \). We ask a little more than this, namely that \( \tilde{\phi}(k) \) is analytic\(^1\) and so expressible as a power series:

\[
\tilde{\phi}(k) = \sum_{\vec{n}=0}^{\infty} \tilde{\phi}_{\vec{n}} k^{\vec{n}} \tag{D.1}
\]

where \( \vec{n} \) is a multi-index, \( \vec{n} = (n_0, n_1, n_2, n_3) \in \mathbb{N} \), and \( k^{\vec{n}} \) is a monomial of the type \( k^{\vec{n}} = \prod_{\mu=0}^{3} k^{\mu} \).

In a classical theory with Poincaré -like symmetries, we could argue that to the field \( \phi(x) \) to be scalar also \( \tilde{\phi}(k) \) has to be a scalar, and in its power series expansion we have only monomial with contracted indices. Here we don’t pose this limitation, and follow our calculations for generic functions.

In chapter two we defined a space-time integral to be equal to the zero-component of the Fourier transform:

\[
\int d^4 \hat{x} \phi(\hat{x}) = \tilde{\phi}(0) \tag{D.2}
\]

that, expressing \( \tilde{\phi} \) as a power series, becomes:

\[
\int d^4 \hat{x} \phi(\hat{x}) = \tilde{\phi}_{\vec{0}} \tag{D.3}
\]

So, a function (of the class above-defined) with null space-time integral has also a power series without zeroth-order term:

\[
\int d^4 \hat{x} \phi(\hat{x}) = 0 \implies \tilde{\phi}(k) = \sum_{\vec{n}=1}^{\infty} \tilde{\phi}_{\vec{n}} k^{\vec{n}} \tag{D.4}
\]

\(^1\) remember that for real functions analyticity condition is less restrictive than for complex functions.
where, in the last equation, $\vec{n} = 1$ in the sum indicates that the latter starts with vectors with at least one non-zero component. The last condition implies in turn that in the power series appear only terms of the first order or above, and so it can be written as:

$$\tilde{\phi}(k) = k_0 \sum_{\vec{n}=0}^{\infty} \tilde{\psi}_0^\vec{n} k^{\vec{n}} + k_i \sum_{\vec{n}=0}^{\infty} \tilde{\psi}_i^\vec{n} k^{\vec{n}} \quad (D.5)$$

with, as usual, $i = 1, 2, 3$ spatial index. It is clear now that anti-transforming back this expression we obtain the identity:

$$\phi(\hat{x}) = P_\mu \psi^\mu(\hat{x}) \quad (D.6)$$

To remove the doubt about the existence of the class of functions we need, we remember that Hermite polynomials have the properties we ask for our Fourier transform $\tilde{\phi}$, and they constitutes a complete basis for square integrable functions space.

\footnote{we are lowering the degree of monomials in the series, so we expect the anti-transformation process to be possible.}
Appendix E

Local commutator for generic functions

A key-step in the demonstration of Noether’s theorem for interacting field theories is to calculate the relation of a product of functions with the same product with reversed order:

\[ f(\hat{x})g(\hat{x}) = G(g(\hat{x})f(\hat{x})) \]  \hspace{1cm} (E.1)

where \( G \) is an arbitrary functional we want to derive here. If we calculate this commutator under a space-time integral, the result is easily obtainable with the aid of Fourier transform:

\[
\int d^4 xd^4 x \left( e^{-3\lambda P_0} g(x) \right) f(x) = \int d^4 x d^4 q \, \tilde{f}(k) \tilde{g}(q) \delta(k_0 + q_0) \delta^3(\vec{k} + e^{-\lambda k_0} \vec{q} + i \lambda q_0) \\
\hspace{1cm} \int d^4 x d^4 q e^{-3\lambda P_0} \tilde{f}(k) \tilde{g}(q) \delta(k_0 + q_0) \delta^3(e^{-\lambda q_0} \vec{k} + \vec{q}) \\
\hspace{1cm} \int d^4 x d^4 q f(k) e^{-3\lambda P_0} g(q) \delta(k_0 + q_0) \delta^3(e^{-\lambda q_0} \vec{k} + \vec{q}) \\
\hspace{1cm} \int d^4 x f(x) \left( e^{-3\lambda P_0} g(x) \right) f(x) \hspace{1cm} (E.2)
\]

The same calculation carried out locally is much more complex; as usual we start from the property of plane waves, and then extend by linearity:
\[ e^{i\vec{k} \cdot \vec{x} - ik_0 x_0} e^{i \vec{q} \cdot \vec{x} - iq_0 x_0} - e^{(e^{i\lambda k_0} - 1) \vec{q} \cdot \vec{x} - iq_0 x_0} e^{i\lambda \vec{k} \cdot \vec{x} - ik_0 x_0} \]
\[ = e^{i(e^{i\lambda k_0} - 1) \vec{q} \cdot \vec{x} e^{i\lambda \vec{k} \cdot \vec{x} - ik_0 x_0} e^{i\lambda k_0} \vec{x} - iq_0 x_0} e^{i\lambda \vec{k} \cdot \vec{x} e^{i\lambda k_0} \vec{x} - ik_0 x_0} \]
\[ = e^{i(e^{i\lambda k_0} - 1) \vec{q} \cdot \vec{x} e^{i(1 - e^{i\lambda k_0}) \vec{k} \cdot \vec{x}} e^{i\lambda \vec{k} \cdot \vec{x} - ik_0 x_0} e^{i\lambda k_0} \vec{x} - iq_0 x_0} e^{i\lambda \vec{k} \cdot \vec{x} e^{i\lambda k_0} \vec{x} - ik_0 x_0} \]
\[ = e^{i\vec{q} \cdot \vec{P} e^{i\lambda k_0} \vec{x} - iq_0 x_0} e^{i\lambda \vec{k} \cdot \vec{x} e^{i\lambda k_0} \vec{x} - ik_0 x_0} \]
\[ = e^{i\vec{q} \cdot \vec{P} e^{i\lambda k_0} \vec{x} - iq_0 x_0} e^{i\lambda \vec{k} \cdot \vec{x} e^{i\lambda k_0} \vec{x} - ik_0 x_0} \]  
(E.4)

where we defined the operators
\[ \rho(P_0) = e^{-\lambda P_0} - 1 \]  
(E.5)

\[ \left[ \vec{P}, \rho(P_0) \right] = \vec{P} \otimes \rho(P_0) - \rho(P_0) \otimes \vec{P} \]  
(E.6)

and the exponential \( e^{i\vec{q} \cdot \vec{P} \rho(P_0)} \) has to be expressed as a power series in which coordinates \( \vec{x} \) are always to the left of spatial momenta.

Finally by linearity of Fourier transforms we have:
\[ f(\hat{x}) g(\hat{x}) = e^{i\hat{q} \cdot \vec{P} \rho(P_0)} g(\hat{x}) \]  
(E.7)

We can use the explicit expression of the local commutator to calculate the difference between it and the “global” one, used in the derivation of the equation of motion; being the calculation a little cumbersome we content ourselves of the first order terms:

\[ e^{i\vec{q} \cdot \vec{P} \rho(P_0)} - e^{-3\lambda P_0} \otimes 1 \approx -i\lambda \vec{x} \cdot \left[ \vec{P}, P_0 \right] \otimes 1 + 3\lambda P_0 \otimes 1 \]  
(E.8)

We are doing a first order analysis, and there is a factor \( \lambda \) multiplying each of the terms of the last expression, so we can consider trivial the coproducts of momenta (they would introduce second order terms):

\[ -i\lambda \vec{x} \cdot \vec{P} \otimes P_0 = P_0 \left( -i\lambda \vec{x} \cdot \vec{P} \otimes 1 \right) + i\lambda \vec{x} \cdot P_0 \vec{P} \otimes 1 \]  
(E.9)

\[ i\lambda \vec{x} \cdot P_0 \otimes \vec{P} = \vec{P} \cdot (i\lambda \vec{x} P_0 \otimes 1) - i\lambda \vec{P} \cdot \vec{x} P_0 \otimes 1 = \vec{P} \cdot (i\lambda \vec{x} P_0 \otimes 1) - 3\lambda P_0 \otimes 1 - i\lambda \vec{x} \cdot \vec{P} P_0 \otimes 1 \]  
(E.10)

that gives us, resumming all the terms:

\[ e^{i\vec{q} \cdot \vec{P} \rho(P_0)} - e^{-3\lambda P_0} \otimes 1 \approx P_0 \left( -i\lambda \vec{x} \cdot \vec{P} \otimes 1 \right) + \vec{P} \cdot (i\lambda \vec{x} P_0 \otimes 1) \]  
(E.11)
This is a confirmation of the “rule” derived in the last appendix; we proved that the difference between “global” and “local” commutator - which has null space-time integral - can be expressed as a four-divergence (at least up to first order terms):

\[
e^{i\vec{x} \cdot \vec{P}} [P_\rho (P_0)] \otimes e^{-3\lambda P_0} \otimes 1 \approx P_\mu J^\mu \quad (E.12)
\]

The imaginary unit in the last term of (E.11) might suggest an imaginary part of the charge, but we note that the operator \( i\vec{x} \cdot \vec{P} \) is real and parity invariant in momentum space (it is \( k \cdot \nabla_k \)), so it doesn’t change a function property under conjugation.
Appendix F

Global divergence theorem

In commutative space-time physics one of the most used theorems, in every background, is the Gauss - or the divergence - theorem. It however exploit local, infinitesimal properties of spacetime that we don’t have (we don’t have, for example, arbitrarily small space-time volumes), so we cannot import it in k-Minkowski without troubles\footnote{Although we can argue it should be available in the spatial-only sector, because of the peculiarity of \eqref{3.2}, that make the pure space commutative.}.

We want however to take as much as possible of this theorem, and in particular we don’t have problem to demonstrate what in the commutative case is a corollary of Gauss theorem.

For fast-decreasing commutative fields we have:

\[
\lim_{V \to \mathbb{R}^n} \int_V d^n x \nabla \cdot \mathbf{f}(x) = \lim_{S \to \infty} \int_S d^{n-1} x \left( \mathbf{f}(x) \cdot \mathbf{n} \right) = 0 \quad \text{(F.1)}
\]

In k-Minkowski we can work around the problem of not having Gauss theorem, exploiting properties of Fourier transform; for sufficiently regular fields we have:

\[
\int d^4 \tilde{x} P_{\mu} f^\mu = \int d^4 k \left( k_{\mu} \tilde{f}^\mu(k) \right) \delta^4(0) = \left( k_{\mu} \tilde{f}^\mu \right)(0) = 0 \quad \text{(F.2)}
\]

So we can still argue that space-time integrals of divergences are null.
Bibliography


