Statistical Mechanics of continuous spin models and applications to nonlinear optics in disordered media

Candidate: Alessia Marruzzo
Supervisor: Dr. Luca Leuzzi
Sapienza University of Rome - Graduate School Vito Volterra
Optics

1. Random Lasers (Multiple Scattering)

2. Laser theory: disorder in the refractive index, $\epsilon(\vec{r})$, strong couplings with the outside, stronger effects of nonlinearities (and multimode fields). Dynamical equations for the mode amplitudes $\rightarrow$ Graph theory and Cavity Method

3. Quantum Theory of the laser: threshold, photon statistics and linewidth. Two modes?

Presence of many modes: Statistical Mechanics Approach

XY and $p$-clock models with nonlinear interactions

- Graph theory: Bethe and Erdős Rényi random graphs
- Detailed analysis of the convergence of the $p$-clock model as $p$ increases:
  - Replica Symmetric Cavity Method
  - 1-step Replica Symmetry Breaking Cavity Method

Electric field of the different phases on instances of Random Factor Graphs

Inference on the XY model
Optics

1  Random Lasers (Multiple Scattering)

2  Laser theory: disorder in the refractive index, $\epsilon(\vec{r})$, strong couplings with the outside, stronger effects of nonlinearities (and multimode fields). Dynamical equations for the mode amplitudes $\Rightarrow$ Graph theory and Cavity Method

8  Quantum Theory of the laser: threshold, photon statistics and linewidth. Two modes?

Presence of many modes: Statistical Mechanics Approach

XY and $p$-clock models with nonlinear interactions

- Graph theory: Bethe and Erdős Rényi random graphs
- Detailed analysis of the convergence of the $p$-clock model as $p$ increases:
  - Replica Symmetric Cavity Method
  - 1-step Replica Symmetry Breaking Cavity Method
- Electric field of the different phases on instances of Random Factor Graphs
- Inference on the XY model
Optics

1. Random Lasers (Multiple Scattering)

2. Laser theory: disorder in the refractive index, $\epsilon(\vec{r})$, strong couplings with the outside, stronger effects of nonlinearities (and multimode fields). Dynamical equations for the mode amplitudes $\Rightarrow$ Graph theory and Cavity Method

3. Quantum Theory of the laser: threshold, photon statistics and linewidth. Two modes?
Optics

1. Random Lasers (Multiple Scattering)

2. Laser theory: disorder in the refractive index, $\epsilon(\vec{r})$, strong couplings with the outside, stronger effects of nonlinearities (and multimode fields). Dynamical equations for the mode amplitudes $\Rightarrow$ Graph theory and Cavity Method

3. Quantum Theory of the laser: threshold, photon statistics and linewidth. Two modes?

Presence of many modes: Statistical Mechanics Approach

XY and $p$-clock models with nonlinear interactions

1. Graph theory: Bethe and Erdös Rényi random graphs

2. Detailed analysis of the convergence of the $p$-clock model as $p$ increases:
   - Replica Symmetric Cavity Method
   - 1-step Replica Symmetry Breaking Cavity Method

3. Electric field of the different phases on instances of Random Factor Graphs

4. Inference on the XY model
Optics

1 Random Lasers (Multiple Scattering)

2 Laser theory: disorder in the refractive index, $\epsilon(\vec{r})$, strong couplings with the outside, stronger effects of nonlinearities (and multimode fields). Dynamical equations for the mode amplitudes $\Rightarrow$ Graph theory and Cavity Method

3 Quantum Theory of the laser: threshold, photon statistics and linewidth. Two modes?

Presence of many modes:
Statistical Mechanics Approach

XY and $p$-clock models with nonlinear interactions

1 Graph theory: Bethe and Erdös Rényi random graphs

2 Detailed analysis of the convergence of the $p$-clock model as $p$ increases:
   - Replica Symmetric Cavity Method
   - 1-step Replica Symmetry Breaking Cavity Method

3 Electric field of the different phases on instances of Random Factor Graphs

4 Inference on the XY model
Optics

1. Random Lasers (Multiple Scattering)

2. Laser theory: disorder in the refractive index, $\epsilon(\vec{r})$, strong couplings with the outside, stronger effects of nonlinearities (and multimode fields). Dynamical equations for the mode amplitudes $\Rightarrow$ Graph theory and Cavity Method

3. Quantum Theory of the laser: threshold, photon statistics and linewidth. Two modes?

Presence of many modes: Statistical Mechanics Approach

XY and $p$-clock models with nonlinear interactions

1. Graph theory: Bethe and Erdös Rényi random graphs

2. Detailed analysis of the convergence of the $p$-clock model as $p$ increases:
   - Replica Symmetric Cavity Method
   - 1-step Replica Symmetry Breaking Cavity Method

3. Electric field of the different phases on instances of Random Factor Graphs

4. Inference on the XY model
**Optics**

1. **Random Lasers** *(Multiple Scattering)*

2. Laser theory: disorder in the refractive index, $\epsilon(\vec{r})$, strong couplings with the outside, stronger effects of nonlinearities (and **multimode fields**). Dynamical equations for the mode amplitudes $\Rightarrow$ Graph theory and Cavity Method

3. Quantum Theory of the laser: threshold, photon statistics and linewidth. **Two modes?**

**Presence of many modes:** **Statistical Mechanics Approach**

- $XY$ and $p$-clock models with nonlinear interactions

  1. Graph theory: Bethe and Erdös Rényi random graphs
  2. Detailed analysis of the convergence of the $p$-clock model as $p$ increases:
     - Replica Symmetric Cavity Method
     - 1-step Replica Symmetry Breaking Cavity Method
  3. Electric field of the different phases on instances of Random Factor Graphs

- Inference on the $XY$ model
Optics

1. Random Lasers (Multiple Scattering)

2. Laser theory: disorder in the refractive index, $\epsilon(\vec{r})$, strong couplings with the outside, stronger effects of nonlinearities (and multimode fields). Dynamical equations for the mode amplitudes $\Rightarrow$ Graph theory and Cavity Method

3. Quantum Theory of the laser: threshold, photon statistics and linewidth. Two modes?

Presence of many modes: Statistical Mechanics Approach

XY and $p$-clock models with nonlinear interactions

4. Graph theory: Bethe and Erdös Rényi random graphs

5. Detailed analysis of the convergence of the $p$-clock model as $p$ increases:
   - Replica Symmetric Cavity Method
   - 1-step Replica Symmetry Breaking Cavity Method

6. Electric field of the different phases on instances of Random Factor Graphs

7. Inference on the XY model
Lasers: gain medium and a cavity that provides feedback → disorder is detrimental

Letokhov, V *Sov. Phys. JETP, 26 (1968)*: if the multiple scattering is strong enough the time light stays in the gain medium is increased enhancing stimulated emission and eventually leading to laser action.

Lasers that use disorder induced scattering as feedback mechanism for light amplification are known as Random Lasers (RLs).
In multiple scattering processes interference effects occur and define the optical modes with a certain central frequency and a spatial profile.

**Figure:** Reprinted from Cao et al. *Phys. Rev. Lett.*, 82, 2278 (1999): Spectra of emission from ZnO powder when the excitation intensity is (from bottom to top then from left to right) 400, 562, 763, 875, and 1387 kW/cm². First experiment showing narrow peaks appear on top of a global narrowing. As the pump increases, new peaks appear.
Experimental features and applications

- Cheaper and simpler technology achieving micrometer size,

- Broad angular distribution of emitted light and low spatial coherence suitable for display and imaging applications,

- The output can be controlled through a Spatial Light Modulator,

For high enough pumps RLs can sustain a large number of modes showing a complex behavior in the spectral response: the laser can have a different spectrum each time is excited and the narrow emission peaks appear at different frequencies,


We adopt a Statistical Mechanics formulation and, because of the complex mode dynamics, we will use results obtained in the theory of spin glasses.
Dynamics of the slow amplitude modes

Semiclassical laser theory:

starting from the Cold-Cavity modes (strong coupling with the outside) adding interaction with the gain medium (up to third order in the interaction strength)

mode lifetimes $\gg$ characteristic time of atomic pump and losses

semiclassical limit $\rightarrow$ equations for the modes alone:

$$E(r, t) = \sum_k a_k(t)e^{-i\omega_k t}E_k(r, t) + c.c.$$  

Dynamics of the Slow Amplitudes $a_k$ for Open Cavities:


$$\frac{da_k(t)}{dt} = -\sum_{k_1|FM(k, k_1)} [\tilde{\Gamma}_{kk_1} + M^{(2)}_{kk_1}]a_{k_1}(t) - \sum_{\vec{k}|FM(k, \vec{k})} M^{(4)}_{k\vec{k}}a_{k_1}(t)a^{*}_{k_2}(t)a_{k_3}(t) + F_k$$  

$\vec{k} \equiv \{k_1, k_2, k_3\}$
Couplings:

\[ |M^{(2m)}_{kk}| \propto \int dr \rho(r) g^L_k(r) g^R_{k_1}(r) g^{R*}_{k_2}(r) \cdots g^{R}_{k_{2m-1}}(r) \]

Frequency Matching (FM) Condition:

\[
FM(k, k_1) \equiv |\omega_k - \omega_{k_1}| \lesssim \delta \omega \rightarrow \delta_{kk_1} \Delta \omega \gg \delta \omega \quad (1)
\]

\[
FM(k, \vec{k}) \equiv |\omega_k - \omega_{k_1} + \omega_{k_2} - \omega_{k_3}| \lesssim \delta \omega
\]

White Gaussian Noise → generalized temperature:

\[
\langle F_k(t) F_{k'}(t') \rangle = 2 T \delta_{kk'} \delta(t - t')
\]

Mode-Locking Transition
Passive mode-locking (saturable absorber) in closed cavity lasers:

\[
\dot{a}_k = (g_k - l_k + i\phi_k) a_k(T) + (i\gamma_k - \gamma_s) \sum_{j-m+l-k=0} a_j(T) a^*_m(T) a_l(T)
\]

**Gain Saturation:** in lasers the gain depends on the intensity of the modes decreasing as the modes intensify

\[ g = \frac{g_0}{1 + \frac{\mathcal{E}}{E_{\text{sat}}}} \quad \mathcal{E} = \sum_k a_k a_k^* \]

We introduce gain saturation as a *spherical constraint* (Gordon and Fisher, Phys. Rev. Lett., 89, 103901 (2002)):

\[ \frac{d}{dt} \left( \sum_k a_k a_k^* \right) = 0 \]

Mode-Locking transition as a thermodynamic phase transition taking into account the Frequency-Matching Condition: beyond the narrow-band approximation

\[ \mathcal{H}_I \sim 0 \rightarrow \text{Langevin equations with effective temperature } T: \text{ the steady-state distribution of the } a_k \text{ is of Gibbs form} \]

\[ \rho(a_1, \ldots, a_N) = \frac{e^{-\frac{\mathcal{H}_R}{T}}}{Z} \]
**Gain Saturation:** in lasers the gain depends on the intensity of the modes decreasing as the modes intensify

\[
g = \frac{g_0}{1 + \mathcal{E}/E_{\text{sat}}} \quad \mathcal{E} = \sum_k a_k a_k^*
\]

We introduce gain saturation as a *spherical constraint* \((\text{Gordon and Fisher, Phys. Rev. Lett., 89, 103901 (2002)})\):

\[
d\left(\sum_k a_k a_k^*\right)/(dt) = 0
\]

Mode-Locking transition as a thermodynamic phase transition taking into account the Frequency-Matching Condition: beyond the narrow-band approximation

\[
\mathcal{H}_I \sim 0 \rightarrow \text{Langevin equations with effective temperature } T: \text{ the steady-state distribution of the } a_k \text{ is of Gibbs form }
\]

\[
\rho(a_1, \ldots, a_N) = \frac{e^{-\frac{\mathcal{H}_R}{T}}}{Z}
\]
- \( a_k \equiv A_k e^{i\phi_k} \rightarrow \) interests on the correlation among phases: *quench amplitude approximation* where the \( A_k \) are considered as constant Antenucci *et al* Phys. Rev A, 91, 043811 (2015)

\[
H = - \sum_{jklm} J_{jklm} \cos (\phi_j - \phi_k + \phi_l - \phi_m)
\]

Hamiltonian of the XY model: \( \sigma = \{\cos \phi, \sin \phi\} \)

For numerical reason \( \rightarrow \) \( p \)-clock model: \( \phi \) can take \( p \) values, equispaced in radians by \( 2\pi/p \):

\[
\phi_a = \frac{2\pi}{p} a; \quad a = 0, 1, \ldots, p - 1
\] (2)

- as first step ordered case: \( \omega_i - \omega_{i+1} = \Delta \omega \gg \delta \omega, J_{jklm} \sim \text{const} \)

Frequency Matching condition \( \rightarrow \) dilution of the graph: the mean connectivity of each node is a constant \( c \ll N \)

\[
P(\sigma) = \frac{1}{Z} \prod_{m=1}^{M} \psi_m (\sigma \partial m)
\] (3)

\( M \equiv \text{total number of quadruplets (function nodes)}; \partial m \text{ spins } m \text{ depends on}; \sigma_i \equiv \text{variable node} \)
\( a_k \equiv A_k e^{i\phi_k} \rightarrow \) interests on the correlation among phases: *quench amplitude approximation* where the \( A_k \) are considered as constant \cite{Antenucci:2015}

\[
H = - \sum_{jklm} J_{jklm} \cos (\phi_j - \phi_k + \phi_l - \phi_m)
\]

Hamiltonian of the XY model: \( \sigma = \{ \cos \phi, \sin \phi \} \)

For numerical reason \( \rightarrow \) \( p \)-clock model: \( \phi \) can take \( p \) values, equispaced in radiants by \( 2\pi/p \):

\[
\phi_a = \frac{2\pi}{p} a; \quad a = 0, 1, \ldots, p - 1
\]

as first step ordered case: \( \omega_i - \omega_{i+1} = \Delta \omega \gg \delta \omega \), \( J_{jklm} \sim \) const

Frequency Matching condition \( \rightarrow \) dilution of the graph: the mean connectivity of each node is a constant \( c \ll N \)

\[
P(\sigma) = \frac{1}{Z} \prod_{m=1}^{M} \psi_m (\sigma \partial_m) \quad (3)
\]

\( M \equiv \) total number of quadruplets (function nodes); \( \partial_m \) spins \( m \) depends on; \( \sigma_i \equiv \) variable node
- $a_k \equiv A_k e^{i\phi_k} \rightarrow$ interests on the correlation among phases: *quench amplitude approximation* where the $A_k$ are considered as constant Antenucci *et al* Phys. Rev A, 91, 043811 (2015)

$$H = - \sum_{jklm} J_{jklm} \cos (\phi_j - \phi_k + \phi_l - \phi_m)$$

Hamiltonian of the XY model: $\sigma = \{\cos \phi, \sin \phi\}$

For numerical reason $\rightarrow$ $p$-clock model: $\phi$ can take $p$ values, equispaced in radians by $2\pi / p$:

$$\phi_a = \frac{2\pi}{p} a; \quad a = 0, 1, \ldots, p - 1 \quad (2)$$

- as first step ordered case: $\omega_i - \omega_{i+1} = \Delta \omega \gg \delta \omega$, $J_{jklm} \sim \text{const}$

Frequency Matching condition $\rightarrow$ dilution of the graph: the mean connectivity of each node is a constant $c \ll N$

$$P(\sigma) = \frac{1}{Z} \prod_{m=1}^{M} \psi_m (\sigma \partial_m) \quad (3)$$

$M \equiv$ total number of quadruplets (function nodes); $\partial m$ spins $m$ depends on; $\sigma_i \equiv$ variable node
Belief Propagation (Sum-Product Algorithm)

Factor graphs yield a graphical representation

- Exact on three-graphs → iterative algorithm on sparse graphs with long \(O(\log N)\) loops
- Cavity Method (CM)
  - On random factor graphs becomes equalities among distributions solve with Population Dynamical Algorithm (PDA) \(\{P^*(\nu), Q^*(\hat{\nu})\}\)

\[
\begin{align*}
\hat{\nu}(\phi) &= \frac{d}{Z_{\text{test}}} \int_{0}^{2\pi} \prod_{l=1}^{k-1} d\phi_l \nu^l(\phi_l) \psi(\phi_1, \ldots, \phi_{k-1}, \phi) ; \\
\nu(\phi) &= \frac{d}{Z_{\text{cav}}} \prod_{m=1}^{c-1} \hat{\nu}^m(\phi) \\
\nu(\phi) &= Z(\phi) / (\int d\phi Z(\phi)) ; \\
\psi(\phi_1, \ldots, \phi_{k-1}, \phi) &= e^{\beta J \cos(\phi_1 - \phi_2 + \phi_3 - \phi)}
\end{align*}
\]
Belief Propagation (Sum-Product Algorithm)

Factor graphs yield a graphical representation

- Exact on three-graphs → iterative algorithm on sparse graphs with long \(O(\log N)\) loops
- Cavity Method (CM)
- On random factor graphs becomes equalities among distributions solve with Population Dynamical Algorithm (PDA) \(\{P^*(\nu), Q^*(\hat{\nu})\}\)

\[
\hat{\nu}(\phi) = \frac{d}{Z_{\text{test}}} \int_0^{2\pi} \prod_{l=1}^{k-1} d\phi_l \, \nu^l(\phi_l) \psi(\phi_1, \ldots, \phi_{k-1}, \phi);
\nu^m(\phi) = \frac{1}{Z_{\text{cav}}} \prod_{m=1}^{c-1} \hat{\nu}^m(\phi);
\nu(\phi) = \frac{Z(\phi)}{\left(\int d\phi Z(\phi)\right)}; \psi(\phi_1, \ldots, \phi_{k-1}, \phi) = e^{\beta J \cos(\phi_1 - \phi_2 + \phi_3 - \phi)}
\]
Limit of $p \to \infty$ on Bethe and Erdös-Rényi (ER) graphs: $T_s$, $T_c$, free-energy $f$ and magnetization $m$

Above a certain values of $J/T$: phase wave solutions

$$\langle \phi(\omega) \rangle = \phi_0 + \phi' \times (\omega - \omega_0)$$

$$\Delta \omega = 2\pi / T_R$$
Failure of the CM assumptions: correlation length ~ loop size

- Gibbs distribution decomposes into an exponential number of pure states: $F^n$ are iid with 
  \[ \rho(F) \propto e^{\beta x (F - F^R)} \]
  messages depend on $n$: $\nu_{i \rightarrow m}^n$; the number of states with $F^n = N\varphi$ goes as $e^{N\Sigma(\varphi)}$; $\Sigma(\varphi)$ is known as the complexity

- $P(\sigma) = \sum_n w_n P^n(\sigma)$

\[ \Rightarrow \text{due to the many states: apply the previous statistical mechanical formalism to this problem with } \nu_{i \rightarrow m}^n \text{ as nodes} \]

Self consistent equation for the distributions $P(\nu)$, of $P(\nu_{i \rightarrow b})$, and $Q(\hat{\nu})$, of $Q(\nu_{b \rightarrow i})$:

\[ P(\nu) = \frac{d}{Z_E} \mathbb{E}_c \int \prod_{b=1}^{c-1} dQ_b(\hat{\nu}_b) \mathbb{I} [\nu = f(\{\hat{\nu}_b\})] [z(\{\hat{\nu}_b\})]^x \quad (4) \]

\[ Q(\hat{\nu}) = \frac{d}{Z_J} \mathbb{E}_J \int \prod_{i=1}^{k-1} dP_i(\nu_i) \mathbb{I} [\hat{\nu} = \hat{f}(\{\nu_i\}, J)] [\hat{z}(\{\nu_i\}, J)]^x \quad (5) \]
Quenched disorder in $J_{ijkl}^{}$

Failure of the CM assumptions: correlation length $\sim$ loop size

- Gibbs distribution decomposes into an exponential number of pure states: $F^n$ are iid with
  $$\rho(F) \propto e^{\beta x (F - F^R)}$$
  messages depend on $n$: $\nu^{n}_{i \rightarrow m}$; the number of states with $F^n = N\varphi$ goes as $e^{N\Sigma(\varphi)}$; $\Sigma(\varphi)$ is known as the complexity

- $P(\sigma) = \sum_n w_n P^n(\sigma)$

$\implies$ due to the many states: apply the previous statistical mechanical formalism to this problem with $\nu^n_{i \rightarrow m}$ as nodes

Self consistent equation for the distributions $P(\nu)$, of $P(\nu_{i \rightarrow b})$, and $Q(\hat{\nu})$, of $Q(\nu_{b \rightarrow i})$:

$$P(\nu) \overset{d}{=} \frac{1}{Z_c} \mathbb{E}_c \int \prod_{b=1}^{c-1} dQ_b(\hat{\nu}_b) \mathbb{I} [\nu = f(\{\hat{\nu}_b\})] [z(\{\hat{\nu}_b\})]^x \quad (4)$$

$$Q(\hat{\nu}) \overset{d}{=} \frac{1}{Z_J} \mathbb{E}_J \int \prod_{i=1}^{k-1} dP_i(\nu_i) \mathbb{I} [\hat{\nu} = \hat{f}(\{\nu_i\}, J)] [\hat{z}(\{\nu_i\}, J)]^x \quad (5)$$
Results: CUDA-C code on multi-gpus
Quantum theory of the laser:

- laser linewidth (phase fluctuations)
- photon count distribution

**Two modes with strong coupling with the outside**

**Contribution of Spontaneous and Stimulated Emission**

Jaynes-Cummings Hamiltonian:

\[
\mathcal{H} = \frac{\hbar}{2} \omega_a \sigma_z + \sum_k \hbar \omega_k \left( a_k^\dagger a_k \right) + \hbar \sum_k g_k \left( \sigma^+ a_k + \sigma^- a_k^\dagger \right)
\]

\[
= \frac{\hbar}{2} \bar{\omega} \sigma_z + \hbar \bar{\omega} \sum_k a_k^\dagger a_k + \frac{\hbar \delta}{2} \sigma_z + \hbar \sum_k \Delta_k a_k^\dagger a_k + \hbar \sum_k g_k \left( \sigma^+ a_k + \sigma^- a_k^\dagger \right)
\]

\[
g_k \propto p_{ab} u_k (r_0), \quad \bar{\omega} = \sum_k \omega_k, \quad \delta = \omega_a - \bar{\omega} \text{ and } \Delta_k = \omega_k - \bar{\omega}
\]
Quantum treatment: two mode laser

Quantum theory of the laser:

- laser linewidth (phase fluctuations)
- photon count distribution

Two modes with strong coupling with the outside

Contribution of Spontaneous and Stimulated Emission

Jaynes-Cummings Hamiltonian:

\[
\mathcal{H} = \frac{\hbar}{2} \omega_\sigma \sigma_z + \sum_k \hbar \omega_k (a_k^\dagger a_k) + \hbar \sum_k g_k \left( \sigma_+ a_k + \sigma_- a_k^\dagger \right)
\]

\[
= \frac{\hbar}{2} \bar{\omega} \sigma_z + \hbar \bar{\omega} \sum_k a_k^\dagger a_k + \frac{\hbar \delta}{2} \sigma_z + \hbar \sum_k \Delta_k a_k^\dagger a_k + \hbar \sum_k g_k \left( \sigma_+ a_k + \sigma_- a_k^\dagger \right)
\]

\[g_k \propto p_{ab} u_k(r_0), \quad \bar{\omega} = \sum_k \omega_k, \quad \delta = \omega_a - \bar{\omega} \text{ and } \Delta_k = \omega_k - \bar{\omega}\]
Two modes, $\Delta k = 0$

Density matrix, $\rho(t)$, describes the radiation field
Losses:

$$
\dot{\rho} = \sum_{\lambda, \lambda'} \Gamma_{\lambda, \lambda'} (2a_\lambda \rho a_{\lambda'}^\dagger - \rho a_{\lambda'} a_\lambda - a_\lambda^\dagger a_{\lambda'} \rho)
$$

$\Delta k = 0$:

- $\rho(t + \tau)$ from interaction with the active medium proceeds as for the case of one mode:

$$
a \rightarrow \alpha = (g_1 a_1 + g_2 a_2)/g \text{ where } g = \sqrt{g_1^2 + g_2^2}
$$

One defines: $\beta = (g_2 a_1 - g_1 a_2)/g$, $[\alpha, \beta^\dagger] = 0$

- Non diagonal term, $\Gamma_{12}$, increases the decay constant of mode $\alpha$:

$$
p(n_\alpha + 1) C_1 - p(n_\alpha) \frac{A}{1 + \delta_\gamma^2 + 4g_\gamma^2(n_\alpha + 1)} = 0
$$

Here, $A = 2 r_a g_\gamma^2$, $g_\gamma = g/\gamma$ and

$$
C_1 = (g_1^2 \Gamma_2 + g_2^2 \Gamma_2 + 2g_1 g_2 + 2g_1 g_2 \Gamma)/(g^2)
$$

Density matrix, $\rho(t)$, describes the radiation field

Losses:

$$\dot{\rho} = \sum_{\lambda, \lambda'} \Gamma_{\lambda, \lambda'} \left( 2a_\lambda \rho a_\lambda^\dagger - \rho a_\lambda^\dagger a_{\lambda'} - a_\lambda^\dagger a_{\lambda'} \rho \right)$$

$\Delta k = 0$:

- $\rho(t + \tau)$ from interaction with the active medium proceeds as for the case of one mode:

$$a \rightarrow \alpha = (g_1 a_1 + g_2 a_2)/g \quad \text{where} \quad g = \sqrt{g_1^2 + g_2^2}$$

One defines: $\beta = (g_2 a_1 - g_1 a_2)/g$, $[\alpha, \beta^\dagger] = 0$

- Non diagonal term, $\Gamma_{12}$, increases the decay constant of mode $\alpha$:

$$p(n_\alpha + 1)C_1 - p(n_\alpha) \frac{A}{1 + \delta_\gamma^2 + 4g_\gamma^2(n_\alpha + 1)} = 0$$

Here, $A = 2r_\alpha g_\gamma^2$, $g_\gamma = g/\gamma$ and

$$C_1 = \frac{g_1^2 \Gamma_2 + g_2^2 \Gamma_2 + 2g_1 g_2 + 2g_1 g_2 \Gamma}{(g^2)}$$

\[ C_2 \langle n_\beta \rangle = 0 \]
\[ C_1 \langle n_\alpha \rangle = \sum_{n_\alpha} p(n_\alpha) \frac{A(n_\alpha + 1)}{1 + \delta^2_\gamma + 4g^2_\gamma(n_\alpha + 1)} \]

\[ \Delta_k \neq 0: \quad B = \sum_k \Delta_k \alpha_k^{\dagger} \alpha_k, \quad [\alpha, B] \neq 0 \]

\[ B | n_\alpha, n_\beta \rangle = \frac{1}{g} \left[ \Delta_d (n_\alpha - n_\beta) | n_\alpha, n_\beta \rangle - \Delta_{nd} (\sqrt{(n_\alpha + 1)n_\beta} | n_\alpha + 1, n_\beta - 1) + \sqrt{(n_\beta + 1)n_\alpha} | n_\alpha - 1, n_\beta + 1) \right] \]

\[ \Delta_d = \Delta / g^2 (g_2^2 - g_1^2) \]; \[ \Delta_{nd} = \Delta / g^2 2g_1g_2 \]

\[ C_1 \langle n_\alpha \rangle = \sum_{n_\alpha} p(n_\alpha) \left\{ A(1 + n_\alpha) \left[ 1 - \delta^2_\gamma - 4g^2_\gamma (1 + n_\alpha) + 2\delta_\gamma \frac{g_2^2 - g_1^2}{g^2} \frac{\Delta}{\gamma} \right. \right. \]
\[ \left. \left. - \left(1 + 12\frac{g_1^2g_2^2}{g^4}\right) \left(\frac{\Delta}{\gamma}\right)^2 \right] - A \frac{4g_1^2g_2^2}{g^4} \frac{\gamma^2 + 4g^2_\gamma}{g^2} (\langle n_\alpha \rangle - \langle n_\beta \rangle) \left(\frac{\Delta}{\gamma}\right)^2 \right\} \]

\[ C_2 \langle n_\beta \rangle = 4 \frac{g_1^2g_2^2}{g^4} A \left[ 3(\langle n_\alpha \rangle + 1) + (\langle n_\alpha \rangle - \langle n_\beta \rangle) \frac{g^2 + \gamma^2}{g^2} \right] \left(\frac{\Delta}{\gamma}\right)^2 \]
\[ C_2 \langle n_\beta \rangle = 0 \]
\[ C_1 \langle n_\alpha \rangle = \sum_{n_\alpha} p(n_\alpha) \frac{A(n_\alpha + 1)}{1 + \delta_\gamma^2 + 4g_\gamma^2(n_\alpha + 1)} \]

\[ \Delta_k \neq 0: B = \sum_k \Delta_k \alpha_k^\dagger \alpha_k, [\alpha, B] \neq 0 \]

\[ B | n_\alpha, n_\beta \rangle = \frac{1}{g} \left[ \Delta_d (n_\alpha - n_\beta) | n_\alpha, n_\beta \rangle - \Delta_{nd} (\sqrt{(n_\alpha + 1)n_\beta} | n_\alpha + 1, n_\beta - 1 \rangle + \sqrt{(n_\beta + 1)n_\alpha} | n_\alpha - 1, n_\beta + 1 \rangle) \right] \]

\[ \Delta_d = \Delta / g^2 (g_2^2 - g_1^2); \Delta_{nd} = \Delta / g^2 2g_1g_2 \]

\[ C_1 \langle n_\alpha \rangle = \sum_{n_\alpha} p(n_\alpha) \left\{ A(1 + n_\alpha) \left[ 1 - \delta_\gamma^2 - 4g_\gamma^2 (1 + n_\alpha) + 2\delta_\gamma \frac{g_2^2 - g_1^2}{g^2} \frac{\Delta}{\gamma} - (1 + 12 \frac{g_1^2g_2^2}{g^4}) \left( \frac{\Delta}{\gamma} \right)^2 \right] \right\} - A \frac{4g_1^2g_2^2}{g^4} \left( \frac{\gamma^2 + 4g^2}{g^2} \right) (\langle n_\alpha \rangle - \langle n_\beta \rangle) \left( \frac{\Delta}{\gamma} \right)^2 \]

\[ C_2 \langle n_\beta \rangle = 4 \frac{g_1^2g_2^2}{g^4} A \left[ 3 (\langle n_\alpha \rangle + 1) + (\langle n_\alpha \rangle - \langle n_\beta \rangle) \frac{g^2 + \gamma^2}{g^2} \right] \left( \frac{\Delta}{\gamma} \right)^2 \]
Conclusions and Prospectives

1. Statistical Mechanical formulation to describe laser behavior in multimode regime: problem is mapped on factor graphs → Cavity Method and 1-Step RSB
   - $p$-clock model introduced for numerical algorithms, detailed analysis on the convergence of $p \rightarrow \infty$ for non-linear interactions
   - On instance of graphs with FMC: $\langle \phi \rangle = \phi_0 + \phi' \times (\omega - \omega_0)$ for both continuous and discrete frequency distributions; melting of light pulses?

2. Quantum theory of two overlapping modes:
   - Attempt to go beyond the limit $\omega_1 = \omega_2 \rightarrow$ both $\alpha$ and $\beta$ modes show a laser transition (under analysis)

3. Inverse Problem: inferring the couplings $J$
   - Pseudolikelihood method: network of interactions among $XY$ spins (linear and non-linear case)
   - through experimental data $\rightarrow$ spatial distribution of the modes

Thank you
Conclusions and Prospectives

1. Statistical Mechanical formulation to describe laser behavior in multimode regime: problem is mapped on factor graphs $\rightarrow$ Cavity Method and 1-Step RSB
   - $p$-clock model introduced for numerical algorithms, detailed analysis on the convergence of $p \rightarrow \infty$ for non-linear interactions
   - On instance of graphs with FMC: $\langle \phi \rangle = \phi_0 + \phi' \times (\omega - \omega_0)$ for both continuous and discrete frequency distributions; melting of light pulses?

2. Quantum theory of two overlapping modes:
   - Attempt to go beyond the limit $\omega_1 = \omega_2 \rightarrow$ both $\alpha$ and $\beta$ modes show a laser transition (under analysis)

3. Inverse Problem: inferring the couplings $J$
   - Pseudolikelihood method: network of interactions among $XY$ spins (linear and non-linear case)
   - through experimental data $\rightarrow$ spatial distribution of the modes

Thank you