Statistical Mechanics of continuous spin models and applications to nonlinear optics in disordered media

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OUTLINE

1. What are Random Lasers (RLs): some review and applications;

   The XY and spherical models: in what approximation we can study their behaviors as a effective models for interacting modes in RLs;
   Methods:
   - Belief Propagation (BP) equations (the cavity method);
   - Monte-Carlo simulations.

2. What I have been doing so far:
   - XY-model with linear interaction on random regular and Erdos-Renyi graphs, non-linear interaction on regular graphs;
   - Frequency distribution of specially designed frequency locked tree-like graphs;

3. What I am going to do:
   - XY model with non-linear interactions on random graphs with quenched disorder;
   - study of the topology of random graphs with the frequencies locked constraint;
   - Monte-Carlo simulations of finite dimension linear models with frequencies locked constraint.
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Lasers → material that provides gain and a cavity: modes that amplify are the eigenmodes of the cavity

Letokhov (1968): multiple scattering could trap light inside the system long enough for the gain to overtake the loss

Tiny Lasers: Zinc Oxide (ZnO) nanoclusters of 10µm thick;

Broad angular distribution → imaging applications;

Cryptography

compact chip-based spectrometer with 0.75 nm resolution around the wavelength of 1,500nm in a 25 − µm-radius structure

Experimental Results

- at low pumping level → single broad spontaneous emission peak;
- when pumping exceeds a threshold → discrete narrow peaks (FWHM 0.2 nm); frequencies of sharp peaks depend on sample position.

Effects of possible interactions among the optical modes?


- By tuning the spatial overlap of the modes, the RL can be prepared in two distinct regimes;
- As the number of interactions increases, the phases of the modes begin to synchronize: mode-locked regime


Phase-transition theory and laser physics:
- laser in multimode regime \((10^2 - 10^9)\) modes, \(\chi^{(3)}\) → interactions among the modes

Lasers are not in thermal equilibrium but often reach a steady state; statistical distribution of the modes is of Gibbs-form with generalized temperature and energy → Mode-Locking transition as an order-disorder phase transition.
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Mode-Locking transition as an order-disorder phase transition.


RL, Hamiltonian Formulation

Maxwell equations in the presence of $P_{NL}(r)$ in a cavity:

\[ \nabla \times H(r) = \epsilon_0 n^2(r) \frac{\partial E(r)}{\partial t} + \frac{\partial P_{NL}(r)}{\partial t} \]  

(1)

\[ \nabla \times E(r) = -\mu_0 \frac{\partial H(r)}{\partial t} \]  

(2)

Admit solutions in the form of superposition of normal modes:

\[ E(r, t) = \Re \left[ \sum_k \sqrt{\omega_k} a_k(t) E_k(r) e^{-i\omega_k t} \right] \]  

(3)

\[ H(r, t) = \Re \left[ \sum_k \sqrt{\omega_k} a_k(t) H_k(r) e^{-i\omega_k t} \right] \]  

(4)

with

\[ \mathcal{E} = \sum_k \omega_k |a_k(t)|^2 \]
Time evolution of the complex amplitude, $a_m(t)$, is described by

$$\frac{d a_m(t)}{dt} = i \frac{\sqrt{\omega_m}}{4} \int_V E_m^* \cdot P_m(r) \, dV$$

(5)

with

$$P_{NL}(r) = \Re \left[ \sum_m \sqrt{\omega_m} a_m(t) P_m(r) e^{-i\omega_m t} \right]$$

$$P_m^\alpha = \sum_{\omega_l} \chi^{(1)}_{\alpha\beta}(\omega_l, \omega_m, r) E_j^\beta(r) \sqrt{\omega_l} a_l(t) + \sum_{\omega_k} \chi^{(3)}_{\alpha\beta\gamma\delta}(\omega_m, \omega_k, \omega_l, -\omega_j, r) E_j^\beta(r) E_k^\gamma(r) E_l^\delta(r) \sqrt{\omega_j \omega_k \omega_l} a_k(t) a_l(t) a_j^*(t)$$

(6)

$$\frac{d a_m(t)}{dt} = \sum_l g_{ml} a_l(t) + \sum_{jkl} g_{mjk\ell} a_j^*(t) a_k(t) a_l(t) + \Gamma_m(t)$$

(7)

$\Gamma_m(t) \to$ spontaneous emission $\langle \Gamma_m(t) \Gamma_l(t') \rangle = \delta_{ml} \delta(t - t') 2 T$
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$$\Gamma_m(t) \rightarrow \text{spontaneous emission}$$

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Hamiltonian formulation

\[ \frac{da_m(t)}{dt} = \sum_l g_{ml} a_l(t) + \sum_{jkl} g_{mjkl} a_j^*(t) a_k(t) a_l(t) + \Gamma_m(t) \]

Non linearity + disorder

In the ordered case:

\[ \frac{\partial a_m}{\partial t} \equiv \dot{a}_m = [g_m - \ell_m + iD_m] a_m + (\gamma - i\delta) \sum_{\omega_j + \omega_k - \omega_l = \omega_m} a_j^* a_k a_l + \Gamma_m(t) \]

HA Haus, Waves and Fields in Optoelectronics, 1984
HAMiLTONiAN FORMULATION

\[
\frac{d a_m(t)}{dt} = \sum_l g_{ml} a_l(t) + \sum_{jkl} g_{mjkl} a^*_j(t) a_k(t) a_l(t) + \Gamma_m(t)
\]

Non linearity + disorder

\[
H = -\Re \left[ \sum_{m} g_{ml} a^*_m a_l + \sum_{mjk} g_{mjkl} a^*_m a^*_j a_k a_l \right]
\]  

(8)
THE XY-MODEL

Hypothesis:
steady-state regime

\[ a(t) = Ae^{i\varphi(t)} \]

A slowly varying with respect to \( \varphi \)

\[ \mathcal{H} = - \sum_{i<j<k<l} J_{ijkl} \cos(\varphi_i + \varphi_j - \varphi_k - \varphi_l) \]

Hamiltonian of the XY-model with a nonlinear interaction term.
In the diluted case, this Hamiltonian is derived in different contexts, such as constraints satisfaction problems in computer science.

XY-model: spins are unitary vector on a plane, the x – y plane:

\[ \sigma_i \equiv (\cos \varphi_i, \sin \varphi_i) \]

Study the XY-model with nonlinear term on random factor graphs with quenched disorder

Tools: Clock-model, Belief Propagation and Population dynamics algorithm.
The \textit{XY}-model

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Belief Propagation (BP) equations

BP equations exact on three-like factor graph:

$$\mu_{x\to f}(\varphi_x) = \frac{1}{z_{cav}} \prod_{h \in \partial x \setminus f} \hat{\mu}_{h\to x}(\varphi_x)$$

$$\hat{\mu}_{f\to x}^{(t)}(\varphi_x) = \frac{1}{z_{test}} \int d\varphi_{\partial f \setminus x} \psi_f(\varphi_{\partial f}) \prod_{y \in \partial f \setminus x} \mu_{y\to f}^{(t)}(\varphi_y)$$

where

$$\psi_f(\varphi_{\partial f}) = e^{\beta J \cos(\varphi_i + \varphi_j - \varphi_k - \varphi_l)}$$

; $z_{cav \setminus test}$ are normalization constants

$\mu_{x\to f}(\varphi) \equiv \mu_{cav} \to$ marginal probability distribution of $x$ where constraint $f$ is not there

$\hat{\mu}_{f\to x}(\varphi_x) \equiv \hat{\mu}_{test} \to$ imagine to add $x$ to constraint $f$ and only to it, $\hat{\mu}_{test}$ is its probability distribution.
Properties of the Solution

\[ f_\beta = \frac{-1}{\beta} \left( \log z^{(s)} + \frac{c}{k} \log z^{(c)} - c \log z^{(l)} \right) \]

where

\[ z^{(s)} = \int d\varphi \prod_{i=1}^{c} \hat{\mu}_{\text{test}} (\varphi) \]

and

\[ z^{(c)} = \int d\varphi_1 \ldots \varphi_4 e^{\beta J \cos (\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4)} \prod_{i=1}^{k} \mu_{\text{cav}} (\varphi_i) \]

magnetization \( \rightarrow m_{x-\text{dir}} = \int d\varphi \cos (\varphi) \mu (\varphi) \)

where

\[ \mu_x (\varphi_x) = \frac{1}{z} \prod_{h \in \partial x} \hat{\mu}_{h \to x} (\varphi_x) \equiv \text{probability distribution of variable } x \]

\( z \equiv \text{normalization constant} \)
$p$-Clock Model

Paramagnetic solution of $XY$-BP equations:

$$
\mu(\varphi) = \hat{\mu}(\varphi) = \text{const} = \frac{1}{2\pi}
$$

Solutions $\forall \beta$

In order to find numerical solutions: $p-$clock model hierarchy of discretization of the $XY$-model as a function of integer $p$

$$
\varphi_n = \frac{2\pi n}{p}, \quad n = 0, \ldots, p - 1
$$

$p \rightarrow \infty \equiv XY$-model

In a continuous model $\Rightarrow$ infinitesimal fluctuations with infinitesimal energy cost.
Given $p$, we expect $\beta(p)$ above which the $XY$ and clock model will show different results.

for $p = 2 \rightarrow$ Ising-model used for some preliminary testing results.

$$
\hat{\mu} = \frac{1}{2\pi} (1 + c_1 (-1)^n)
$$

Through BP equations we obtain a self-consistent equation for $c_1$
**p-Clock Model**

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Ising model with 4-body interaction: some results for testing

We compared with the results expected for the Ising model on a Bethe Lattice: regular factor graph $\rightarrow c = \text{const}$

**Spinodal point**

**Critical point**
Some preliminary results

Spinodal point

**Figure:** For fixed connectivity $c = 4 \rightarrow$ in the continuous limit infinitesimal excitations drive the system to the paramagnetic solution

Critical and Spinodal points

**Figure:** For $c = 5 \rightarrow$ order-disordered phase transition present in the continuous limit, for $\beta > \beta_c$ becomes the one with smaller free-energy
Graphs and Factor graphs

What I am going to do:
- frequencies distributions with several values for $c$;
- study the topology of ER factor graphs built from a single graph (only pair interactions);
- MC simulations of a linear model with frequency locked constraint → experiments

**Figure:** Parabolic distribution is a possible frequencies distribution on a tree factor graph with fixed connectivity: $c = 3$ and $k = 4$. 
**Graphs and Factor graphs**

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Using Population Dynamics algorithm, solve the BP equations on Random factor graphs with quenched disorder → experiments

M. Mézard and A. Montanari, *Information, Physics and Computations*

Population dynamics algorithm:
The idea is to approximate the distributions of $\mu$ and $\hat{\mu}$ through $N$ i.i.d. copies → as $N$ grows the empirical distributions should converge to the actual distributions.

Considering

$$\hat{\mu}_{f \rightarrow x}^{(t)}(\varphi_x) = \frac{1}{Z_{test}} \int d\varphi_{\partial f \setminus x} e^{\beta J \cos (\varphi_i + \varphi_j - \varphi_k - \varphi_l)} \prod_{y \in \partial f \setminus x} \mu_{y \rightarrow f}^{(t)}(\varphi_y)$$

two ways of adding disorder:

$J \rightarrow \mathcal{P}(J)$ $J$ randomly varying among the constraints

$e^{\beta J \cos (\varphi_i + \varphi_j - \varphi_k - \varphi_l - \omega)}$, with $\omega$ randomly varying with $p(\omega)$
XY-model

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Thank you for your attention

References:


