Statistical Mechanics of Disordered Systems: Applications in Optics

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What? Multimode Systems

- (Many) Well-defined Modes $a_k$
  
  Space $E_k(r)$  
  Frequency $\omega_k \gg \Delta \omega$

- Space - Time Separation of the Electromagnetic Field

$$E(r, t) = \Re \left[ \sum_k a_k(t) E_k(r) \right] \quad \text{with} \quad a_k(t) \sim \exp(-i\omega_k t)$$
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• Dynamics for the Complex Mode Amplitudes
  • Near Lasing Regime

  \[
  \frac{d a_j}{dt} = \sum_k G_{jk} a_k + \sum_{klm} G_{jklm} a_k a_l^* a_m + \eta_j
  \]
What? Multimode Systems

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  \begin{align*}
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  \text{Frequency} & \quad \omega_k (\gg \Delta \omega)
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  What are the values for $G$s? Hardly known in Most Cases

  General Properties of $G$s $\rightarrow$ Statistical Mechanics
Langevin Dynamics for Complex Amplitudes

\[
\frac{da_j}{dt} = - \frac{d\mathcal{H}}{da_j^*} + \eta_j \quad \text{with} \quad \mathcal{H} \simeq \sum_{jk} G_{jk} a_j^* a_k + \sum_{jklm} G_{jklm} a_j^* a_k a_l^* a_m
\]

Main Working Hypothesis on $G$s:

$\mathcal{H}$ is real $\quad \rightarrow \quad$ Hamiltonian System
What? Hamiltonian Multimode Systems

Langevin Dynamics for Complex Amplitudes

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Main Working Hypothesis on \( G_s \):
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- Stability of the Steady-State
- Spherical Constraint:
  \[ \sum_j |a_j|^2 \equiv \epsilon N \]
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Machinery Well-Tested for Mean Field SML.


What’s Next?
How? RL Mean Field Theory

• All modes are equal \( \rightarrow \) MFT

Space: Extended Modes Frequency: Narrow Bandwidth

\[
H = -\frac{1}{2N} \sum_{sp}^{1,N} J_{jk} a_s a_p^* - \frac{1}{4!N^3} \sum_{spqr}^{1,N} J_{spqr} a_s a_p^* a_q a_r^*, \quad \sum_k |a_k|^2 = \epsilon N.
\]

with i.i.d. \( J_{sp} \) and \( J_{spqr} \)
How? RL Mean Field Theory


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\]

with i.i.d. \( J_{sp} \) and \( J_{spqr} \)

\[
\frac{J_{sp}}{J_0} = (1 - \alpha_0) J_0 \quad \frac{J_{spqr}}{J_0} = \alpha_0 J_0 \\
\frac{J_{sp}^2}{J_0^2} - \frac{J_{sp}^2}{J_0^2} = (1 - \alpha)^2 J^2 \quad \frac{J_{spqr}^2}{J_0^2} - \frac{J_{spqr}^2}{J_0^2} = \alpha^2 J^2
\]

Control Parameters

• Degree of disorder
  \[ R_J = \frac{J}{J_0} \]

• Pumping rate
  \[ \mathcal{P} = \epsilon \sqrt{\beta J_0} \]

• Degree of nonlinearity
  \[ \alpha = \alpha_0 \]
“New Kind” Of Spin: a Complex Amplitude (XY + Spherical)

Order Parameters ($Q_{aa} \equiv 1 \leftrightarrow \text{SC}$)

\[
Q_{ab} = \sum_j \frac{(a_j^a)^* a_j^b}{\epsilon N}
\]
\[
R_{ab} = \sum_j \frac{\Re [a_j^a a_j^b]}{\epsilon N}
\]
\[
T_{ab} = \sum_j \frac{\Im [a_j^a a_j^b]}{\epsilon N}
\]
\[
m_\sigma = \frac{\sqrt{2}}{\epsilon N} \sum_j \Re [a_j]
\]
\[
m_\tau = \frac{\sqrt{2}}{\epsilon N} \sum_j \Im [a_j]
\]
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\[
m_\sigma = \frac{\sqrt{2}}{\epsilon N} \sum_j \Re[a_j] \\
m_\tau = \frac{\sqrt{2}}{\epsilon N} \sum_j \Im[a_j]
\]

Four Phases:

- **CW**: all OPs are zero
- **PLW**: all zero but $R_{aa}$
- **RL**: $m = 0$ but nontrivial $Q_{ab}$
- **SML**: $m \neq 0$
$\alpha = \alpha_0 = 1$
MFT: Phase Diagram (I)

\[ \alpha = \alpha_0 = 1 \]
MFT: Phase Diagram (II)

\[ \alpha = \alpha_0 = 0.4 \]
MFT: Phase Diagram (III)
MFT: Intensity Overlap

Hard to detect the mode phase correlations in Experiments:
What about the Overlap between Intensity Fluctuations?

\[ I_{ab} \equiv \sum_j \frac{\langle |a_j^a|^2 |a_j^b|^2 \rangle - \langle |a_j^a|^2 \rangle \langle |a_j^b|^2 \rangle}{\epsilon^2 N} \]
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\]

In the MFT it holds (at \( m = 0 \))

\[
I_{ab} \equiv 2 \left( Q_{ab}^2 + R_{ab}^2 \right)^2 + 2 \left( Q_{ab}^2 - R_{ab}^2 \right)^2
\]

Replica Symmetry is spontaneously Broken in the Intensity Fluctuations Overlap in RL regime
MFT: Intensity Overlap - Experiments?

Replica Symmetry is spontaneously Broken in the Intensity Fluctuations Overlap in RL regime

Consider the case of Standard Mode Locking Lasers

Space: Extended Modes  Frequency: \( \text{Comb} \ \delta \omega \ll \Delta \omega = 2\pi c/2L \)

\[
a_{k_1}(t) \cdot a_{k_2}^*(t) \cdot a_{k_3}(t) \cdot a_{k_4}^*(t) \sim \exp \left[ i \left( \omega_{k_1} - \omega_{k_2} + \omega_{k_3} - \omega_{k_4} \right) t \right]
\]
Consider the case of Standard Mode Locking Lasers

\[ a_{k_1}(t) \cdot a^*_{k_2}(t) \cdot a_{k_3}(t) \cdot a^*_{k_4}(t) \sim \exp\left[i(\omega_{k_1} - \omega_{k_2} + \omega_{k_3} - \omega_{k_4})t\right] \]

The Hamiltonian is indeed (purely dissipative case)

\[ \mathcal{H} = - \sum_k G_k |a_k|^2 - \frac{\Gamma}{2} \sum_{\text{FMC}(k)} a_{k_1} \cdot a^*_{k_2} \cdot a_{k_3} \cdot a^*_{k_4}, \quad \sum_k |a_k|^2 = \epsilon N \]

with the Frequency Matching Condition

\[ \text{FMC}(k) : \quad |\omega_{k_1} - \omega_{k_2} + \omega_{k_3} - \omega_{k_4}| \lesssim \delta \omega \]

\[ \delta \omega \ll \Delta \omega \quad \rightarrow \quad k_1 - k_2 + k_3 - k_4 = 0 \]
What? SML Beyond Mean Field

Homogeneous Dilution

FMC Dilution
MC simulations: Results (I)

Energy

![Graph showing energy plots with mean field and other data points]
MC simulations: Results (II)

\[ r = \frac{1}{N} \sum_j |a_j| \]
MC simulations: Results (III)

\[ m_x = \frac{1}{N} \sum_j \mathbb{R}[a_j] \]
MC simulations: Results (IV)

What is the origin of the lacking of the $O(2)$ Symmetry Breaking?

Phase Waves
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Phase Waves
MC simulations: Results (IV)

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**Phase Waves**
MC simulations: Results (V)

\[ a_j = |a_j| \exp(i\Phi_j) : \quad E(t | T) = \sum_{j=1}^{N} |a_j(T)| \exp[i(2\pi\omega_j t + \Phi_j(T))], \quad T \gg t \]

**Phase Delay:** \[ \Phi_j \simeq \Phi_0 + \Phi' \omega_j \]
MC simulations: Results (V)

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MC simulations: Results (VI)

Evidence in the Spectra

\[ I(\lambda) \text{ (a.u.)} \]

\[ \lambda \text{ (a.u.)} \]

\[ PC(N) = 1.60(2); \ N = 150 \]

\[ \sigma_g = 3885 \]

\[ \sigma_g = 243 \]

\[ g(\lambda) \]
Further “Surprises” in SML Beyond MFT:

- Metastability Vanishes in the Thermodynamic Limit
- Vanishing Two Point Correlation Functions
- Slow Dynamics of Phase Waves (different slopes as basins of the FEL)
- Power Condensation ($\mathcal{O}(1)$ modes take $\mathcal{O}(N)$ intensity in hyper-diluted systems)
- Synchronous MC Algorithm

Outlook:

- Random Laser Models Beyond MFT
Thanks for the attention

Stvdvm Vrbis (photos by MIB)