Transport properties in non-equilibrium and anomalous systems

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Structure of the talk

Non-equilibrium

- FDR “Violations”
- Entropy Production

Applications in granular materials
Classical fluctuation-dissipation relations (FDR)

**Einstein relation**
\[
\frac{\delta A(t)}{\delta h(0)} = \text{const.} \langle A(t)A(0) \rangle
\]

Valid for a Brownian particle in a fluid at temperature T

Equivalent to the integrated version

\[ \mu = \beta D \]

Mobility (integrated response)

Diffusion coefficient (integrated correlation)

Statistical features of non perturbed (equilibrium) systems

\[ D = \int_0^{+\infty} \langle v(t)v(0) \rangle \, dt \]

Properties of perturbed (non equilibrium) systems

\[ \mu = \beta D \]
Generalized Fluctuation-dissipation relation (GFDR)


\[ x(t) = S^t x(0) \quad \text{Perturbed trajectory:} \quad x'(0) = x(0) + \delta x \]

Evolution operator

if \( \rho(x) \) is non-vanishing and differentiable, phase space distribution function and the system is mixing

\[ \frac{\delta x_i(t)}{\delta x_j(0)} = - \left\langle x_i(t) \left. \frac{\partial \ln \rho(x)}{\partial x_j} \right|_{t=0} \right\rangle \]

Relaxation of an external perturbation

Steady state correlation (unperturbed system)

GFDR \( \rho(x) \propto e^{-\beta \phi(x)} \) \( \rightarrow \) Einstein relation if \( \phi(x) \) is quadratic without couplings
Granular gas model

\[ \dot{v}_i(t) = -\gamma_b v_i + \sqrt{\frac{\gamma_b T_b}{2}} \eta_t \]

\[ v'_1 = v_1 - \frac{1 + r}{2} (v_1 - v_2) \]

\[ v'_2 = v_2 + \frac{1 + r}{2} (v_1 - v_2) \]

Two times scale system

Mean collision time \( \mathcal{T}_C \)

Thermostat time \( \mathcal{T}_b \)

Equilibrium-like regime
\( \mathcal{T}_C > \mathcal{T}_b \)

Homogeneous spatial distribution
Maxwell distribution of velocities

\( T_b = T_g \)

Colliding regime
\( \mathcal{T}_C \ll \mathcal{T}_b \)

Spatial inhomogeneity is present
Non-Maxwellian deviations can occur

\( T_g \leq T_b \)
Perturbation $\tilde{\delta v}_i$ of tracer's velocity (linear response regime)

D. Villamaina, A. Puglisi, A. Vulpiani, Jstat 2008

**Dilute regime**

$$\rho (\{x, v\}) = n^N \prod_{i=1}^{N} p_v(v_i)$$

$$\frac{\delta v_i(t)}{\delta v_i(0)} = - \left< v_i(t) \frac{\partial \ln p_v(v_i)}{\partial v_i} \right|_{t=0}$$

$$\frac{\delta v(t)}{\delta v(0)} = \frac{1}{T_g} \left< v(t)v(0) \right>$$

Einstein relation is restored, also if velocities are non Gaussian

**Strong dissipation regime**

$$\rho (\{x, v\}) \neq n^N \prod_{i=1}^{N} p_v(v_i)$$

$$\frac{\delta v_i(t)}{\delta v_i(0)} = - \left< v_i(t) \frac{\partial \ln p_v(v_i)}{\partial v_i} \right|_{t=0}$$

Single particle factorization fails

**Coupling between different degrees of freedom makes Einstein relation not valid**
What we learn from FDR

The dense case has two levels of irreversibility

- Non equilibrium due to inelasticity
- Memory effects due to recollisions

It is possible to write down a Langevin equation for the velocity of an intruder in the dilute case
Dilute case
A. Sarracino, D. Villamaina, A. Puglisi, Jstat 2010

\[
\begin{aligned}
    m \dot{u}_i &= -\gamma_b u_i + \eta_b \\
    M \dot{V} &= -\gamma_b V + \eta_b \\
\end{aligned}
\]

\[\frac{m}{M} \gg 1\]

Boltzmann equation
Kramers Moyal expansion

\[M \dot{V} = -\Gamma V + \mathcal{E}\]

Effective equation for the tracer
(Linear only in the large mass limit)

Dense case

\[M \dot{V}_t = -\Gamma V_t + \gamma_s \int_{-\infty}^{t} \gamma(t - t') V_{t'} dt' + \mathcal{E}'\]

Memory terms
Colored noise

Dilute limit
\[\gamma_s \to 0\]
\[\mathcal{E}' \to \mathcal{E}\]
Memory as a hidden variable
A. Puglisi, D. Villamaina, EPL 2009

How do you know you have taken enough variables, for it to be markovian?

Onsager-Machlup

\[ M\dot{V}(t) = -\Gamma V(t) + \gamma_s \int_{-\infty}^{t} \gamma(t - t')V(t')dt' + \mathcal{E}' \]

Mapped in a two variables system (in the case of an exponential kernel)

\[
\begin{pmatrix}
\dot{V} \\
\dot{u}
\end{pmatrix} = \mathbb{A} \begin{pmatrix}
V \\
u
\end{pmatrix} + \begin{pmatrix}
\eta_1 \\
\eta_2
\end{pmatrix}
\]

Colored noise and memory term are not present

What is the physical interpretation of the hidden variable?
Restoring equilibrium-like relations

The mapping suggests that $\rho(V, u)$ is a bivariate Gaussian.

$u(x)$ is assumed to be a **local velocity field**.

$u(x)$ is defined on a small cell of diameter **centered in the tracer**.

$$\ln P(V, u) = aV^2 + buV + u^2$$

$$R(t) = a \langle V(t)V(0) \rangle + b \langle V(t)u(0) \rangle$$
In non-equilibrium cases, for large times \( \langle \Sigma_t \rangle \propto t \)

Using the auxiliary variable model it is possible to define

\[
\Sigma_t = \gamma_s \left( \frac{1}{T_0} - \frac{1}{T_1} \right) \int_0^t V(s)u(s)\,ds
\]

The Gallavotti-Cohen theorem is verified

\[
\ln \frac{P(\Sigma_t = x)}{P(\Sigma_t = -x)} = -x
\]
Conclusions

In non-equilibrium steady states, classical fluctuation-dissipation relations are not expected to hold.

In a granular gas model, the **FDR can be restored** by taking into account the coupling with the local velocity field.

The model introduced verifies the Fluctuation theorem.

Memory due to recollisions is the **main source of non-equilibrium** in dense granular gases.
Perspectives

Non-equilibrium

FDR “Violations”

_entropy Production

single-file diffusion

Piston with few particles