Low-dimensional interacting bosons

PhD research project

Serena Cenatiempo

Physics Department
University of Rome “La Sapienza”

Supervisors
Prof. Vincenzo Marinari
Dr. Alessandro Giuliani

Rome, 9 June 2010
Outline

1. Motivations
2. Known results and open problems
3. Sketch of the strategy
4. Perspectives
Motivations

Due to recent experimental realizations of BEC, the theory of ultracold, dilute Bose gases is currently the subject of intensive studies:

- investigation of interacting atoms that exhibit a long range order;
- interesting applications: atom laser and atom optics.

Experiments on gases in flat or elongated traps strongly motivate the study of Bose condensation in low dimensional systems.

Open problem:
From a theoretical point of view, there are very few (quite special) models in which we are able to prove Bose condensation for interacting bosons.
Motivations

Due to recent experimental realizations of BEC, the theory of ultracold, dilute Bose gases is currently the subject of intensive studies:

▶ investigation of interacting atoms that exhibit a long range order;
▶ interesting applications: atom laser and atom optics.

Experiments on gases in flat or elongated traps strongly motivate the study of Bose condensation in low dimensional systems

Open problem:

From a theoretical point of view, there are very few (quite special) models in which we are able to prove Bose condensation for interacting bosons.
Motivations

Due to recent experimental realizations of BEC, the theory of ultracold, dilute Bose gases is currently the subject of intensive studies:

- investigation of interacting atoms that exhibit a long range order;
- interesting applications: atom laser and atom optics.

Experiments on gases in flat or elongated traps strongly motivate the study of Bose condensation **in low dimensional systems**.

Open problem:

From a theoretical point of view, there are very few (quite special) models in which we are able to prove Bose condensation for interacting bosons.
Bogoliubov approximation (1947)

The starting point

Basic Hamiltonian of the problem:

\[ H = \sum_{i=1}^{N} \left( -\frac{\Delta x_i}{2m} - \mu \right) + \lambda \sum_{i<j} v (x_i - x_j) \]

where

- the scene of action is a periodic box in \( \mathbb{R}^3 \) of side size \( L \)
- \( x_i \) is the coordinate of the \( i^{th} \) particle
- \( N \) is the number of bosons
- The pair potential \( \lambda v (x_i - x_j) \) is \( C^\infty \) and short range

We are interested in the bulk limit: \( N \to \infty \) with \( \rho = \text{costant} \).
Bogoliubov approximation (1947)

The starting point

Basic Hamiltonian of the problem in momentum space:
($a_k^+$ and $a_k$ are boson creation/annihilation operators)

$$H = \sum_k \frac{k^2}{2m} a_k^+ a_k + \frac{1}{2V} \sum_{k,q,p} a_{k+p}^+ a_{q-p}^+ a_k a_q \tilde{v}(p)$$
Bogoliubov approximation (1947)

The starting point

Basic Hamiltonian of the problem in momentum space:
(a\(_k^+\) and a\(_k\) are boson creation/annihilation operators)

\[
H = \sum_k \frac{k^2}{2m} a_k^+ a_k + \frac{1}{2V} \sum_{k,q,p} a_{k+p}^+ a_{q-p} a_k a_q \tilde{v}(p)
\]

In the non interacting case:

\[
N_k = \langle a_k^+ a_k \rangle = N \delta_{k,0}
\]
Bogoliubov approximation (1947)

The starting point

Basic Hamiltonian of the problem in momentum space:
(a_k^+ and a_k are boson creation/annihilation operators)

\[
H = \sum_k \frac{k^2}{2m} a_k^+ a_k + \frac{1}{2V} \sum_{k,q,p} a_{k+p}^+ a_{q-p}^+ a_k a_q \tilde{v}(p)
\]

In the non interacting case:

\[
N_k = \langle a_k^+ a_k \rangle = N\delta_{k,0}
\]
**Bogoliubov approximation (1947)**

*The starting point*

Basic Hamiltonian of the problem in momentum space:

\( H = \sum_k \frac{k^2}{2m} a_k^+ a_k + \frac{1}{2V} \sum_{k,q,p} a_{k+p}^+ a_{q-p} a_k a_q \tilde{v}(p) \)

In the non interacting case:

\( N_k = \langle a_k^+ a_k \rangle = N \delta_{k,0} \)
Bogoliubov approximation (1947)

The starting point

Basic Hamiltonian of the problem in momentum space:
\( (a_k^+ \text{ and } a_k \text{ are boson creation/annihilation operators}) \)

\[
H = \sum_k \frac{k^2}{2m} a_k^+ a_k + \frac{1}{2V} \sum_{k,q,p} a_{k+p}^+ a_{q-p} a_k a_q \tilde{v}(p)
\]

In the non interacting case:
\[
N_k = \langle a_k^+ a_k \rangle = N \delta_{k,0}
\]

Condensation hypothesis:
At weak coupling \( \frac{N_{>0}}{N_0} \ll 1 \)
Bogoliubov approximation (1947)

The starting point

Basic Hamiltonian of the problem in momentum space:
($a_k^+$ and $a_k$ are boson creation/annihilation operators)

$$H = \sum_k \frac{k^2}{2m} a_k^+ a_k + \frac{1}{2V} \sum_{k,q,p} a_{k+p}^+ a_{q-p}^+ a_k a_q \tilde{v}(p)$$

By the substitution $a_0^+ = a_0 = \sqrt{N}$:

$$H = \frac{1}{2V} N(N-1) \tilde{v}(0) + \sum_{k \neq 0} \left[ \frac{k^2}{2m} + \frac{N}{V} \tilde{v}(k) \right] a_k^+ a_k$$

$$+ \frac{1}{2V} \sum_{k \neq 0} \tilde{v}(k) N \left[ a_k^+ a_{-k}^+ + a_k a_{-k} \right] + \ldots$$
Bogoliubov approximation (1947)

The starting point

Basic Hamiltonian of the problem in momentum space:
($a_k^+$ and $a_k$ are boson creation/annihilation operators)

$$H = \sum_k \frac{k^2}{2m} a_k^+ a_k + \frac{1}{2V} \sum_{k,q,p} a_{k+p}^+ a_{q-p}^+ a_k a_q \tilde{v}(p)$$

By the substitution $a_0^+ = a_0 = \sqrt{N}$:

$$H_B = \frac{1}{2V} N(N-1) \tilde{v}(0) + \sum_{k \neq 0} \left[ \frac{k^2}{2m} + \frac{N}{V} \tilde{v}(k) \right] a_k^+ a_k$$

$$+ \frac{1}{2V} \sum_{k \neq 0} \tilde{v}(k) N \left[ a_k^+ a_{-k}^+ + a_k a_{-k} \right]$$

Due to the condensation hypothesis the cubic and quartic terms may be expected to be small in comparison with the quadratic ones.
Motivations

Known results and open problems

Sketch of the strategy

Perspectives

Bogoliubov approximation

Low dimensional results

RG approach

Bogoliubov approximation

Results

$H_B$ is a quadratic form in the $a$'s and may be diagonalized:

$$H'_B = E_0(\rho) + \sum_{k \neq 0} \epsilon(k) \ b^+_k b_k$$

- Spectrum:

$$\epsilon(k) = \sqrt{\left(\frac{k^2}{2m}\right)^2 + 2 \frac{k^2}{2m} \rho \tilde{v}(k)} \sim \sqrt{\frac{\rho \tilde{v}(0)}{m} |k|} = v_B |k|$$

- Ground state energy at low density and in the thermodynamic limit:

$$E_0(\rho) = \frac{1}{2} N \rho \tilde{v}(0) - \frac{N}{4 \pi^2 \rho} \int_0^{\infty} dk \ k^2 \left\{ \frac{k^2}{2m} + \rho \tilde{v}(k) - \epsilon(k) \right\}$$

$$\sim 4 \pi N \frac{\rho}{2m} \left( 1 + \frac{128}{15 \sqrt{\pi}} \sqrt{\rho a^3} + \ldots \right)$$

1 The second order correction was calculated by Lee-Huang-Yang (1957). The leading term was proved by Dyson (upper bound, 1957) and Lieb-Yngvason (lower bound, 1998).
Beyond Bogoliubov approximation

- The Bogoliubov approximation assumes that, for sufficiently weak coupling, a condensed state still exists. This assumption is not a priori justified.

- BEC has been proved in the special case of hard core bosons on a lattice at half-filling; estimates on a few correlation functions have been established.

- Lieb, Seiringer and Yngvason (2002) have proved Bose condensation and superfluidity for 3D and 2D bosons in a trap, but only in the Gross-Pitaevskii limit, in which the density scales with the particle number.

- The calculation of the next corrections to Bogoliubov prediction is an open problem even if a few recent papers present partial results\(^2\).

Low dimensional results

\(d=2\)

- Condensation is expected only for zero temperature. (Mermin-Wagner theorem).
- Ground state energy per particle of a dilute, homogeneous, two-dimensional Bose gas in the thermodynamic limit \(^3\): 
  \[ e_0(\rho) \simeq 4\pi \mu \rho |\log \rho a^2|^{-1} \]

\(d=1\)

- In the presence of repulsive interaction, no condensation is expected.
- The reference model is the exactly soluble Lieb-Liniger model 
  \[ H = - \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2m} + \lambda \sum_{i<j} \delta (x_i - x_j) \]
- A complete analysis of the response functions is still lacking, in spite of recent approaches\(^4\).

---

\(^3\) Schick (1971) and Lieb, Yngvason (2001)

\(^4\) Kormos, Mussardo, and Trombettoni (2009)
Low dimensional results

\(d=2\)

- Condensation is expected only for zero temperature. \((\text{Mermin-Wagner theorem})\).
- Ground state energy per particle of a dilute, homogeneous, two-dimensional Bose gas in the thermodynamic limit \(^3\):

\[
e_0(\rho) \simeq 4\pi \mu \rho |\log \rho a^2|^{-1}
\]

\(d=1\)

- In the presence of repulsive interaction, no condensation is expected.
- The reference model is the exactly soluble Lieb-Liniger model

\[
H = - \sum_{i=1}^{N} \frac{\Delta x_i}{2m} + \lambda \sum_{i<j} \delta (x_i - x_j)
\]

- A complete analysis of the response functions is still lacking, in spite of recent approaches\(^4\).

\(^3\) Schick (1971) and Lieb, Yngvason (2001)

\(^4\) Kormos, Mussardo, and Trombettoni (2009)
Renormalization Group approach

*Functional integral representation*

System of N bosons in $\mathbb{R}^3$ interacting with a repulsive pair potential:

$$H = \sum_{i=1}^{N} \left( - \frac{\Delta x_i}{2m} - \mu \right) + 2\lambda \sum_{i<j} \nu(x_i - x_j)$$

The partition function of the system can be expressed as a functional integral:

$$Z = e^{-E_0|\Lambda|} = \int_\Lambda P(d\varphi) e^{-V(\varphi)}$$

- $\varphi^\pm_x$ → creation and the annihilation operators for the bosons
- $\varphi^\pm_x = e^{Ht}\varphi^\pm_x e^{-Ht}$, $x = (t, x)$
- $\Lambda = \left[ -\frac{\beta}{2}, \frac{\beta}{2} \right] \times \left[ -\frac{L}{2}, \frac{L}{2} \right]^3$
- $P(d\varphi)$ complex Gaussian measure
- $V(\varphi) = \lambda \int_\Lambda \nu(x - y) \delta(x_0 - y_0) \varphi^+_x \varphi^-_x \varphi^+_x \varphi^-_x dx dy$

*Singular perturbation theory*
The properties of BEC for 3D bosons, interacting with a repulsive short range potential, at zero temperature and weak coupling, may been explained in terms of an asymptotically free renormalization group flow:\(^5\):

- the two point correlation function has the typical superfluid behaviour at long wavelengths, at least order by order in the running coupling constants:

  \[ S_I(k) \sim (k_0^2 + v^2 k^2)^{-1} \]

  \[ S_F(k) \sim (-i k_0 + k^2 / 2m)^{-1} \]

- an expression for the speed of sound is obtained, whose leading term, when the coupling goes to zero, coincides with the speed of sound in the Bogoliubov model.

---

Renormalization Group approach

Results

The properties of BEC for 3D bosons, interacting with a repulsive short range potential, at zero temperature and weak coupling, may be explained in terms of an asymptotically free renormalization group flow:

- The two point correlation function has the typical superfluid behaviour at long wavelengths, at least order by order in the running coupling constants:
  
  \[
  S_i(k) \simeq \left( k_0^2 + v^2 k^2 \right)^{-1}
  \]
  
  \[
  S_F(k) \simeq (-ik_0 + \frac{k^2}{2m})^{-1}
  \]

- An expression for the speed of sound is obtained, whose leading term, when the coupling goes to zero, coincides with the speed of sound in the Bogoliubov model.

\[\text{Benfatto (1994) and Pistolesi, Castellani, Di Castro, Strinati (1997, 2004)}\]
Renormalization Group approach

Results

The properties of BEC for 3D bosons, interacting with a repulsive short range potential, at zero temperature and weak coupling, may been explained in terms of an asymptotically free renormalization group flow:\(^5\):

- the two point correlation function has the typical superfluid behaviour at long wavelengths, at least order by order in the running coupling constants:

  $$S_i(k) \simeq (k_0^2 + v^2 k^2)^{-1}$$

  $$S_F(k) \simeq (-i k_0 + \frac{k^2}{2m})^{-1}$$

- an expression for the speed of sound is obtained, whose leading term, when the coupling goes to zero, coincides with the speed of sound in the Bogoliubov model.

Renormalization Group approach

Reference papers

**Benfatto**

Rigorous RG techniques (Gallavotti, 1985) have been used to approach the 3D problem; one can prove that the theory is order by order finite in the renormalized coupling constants, with the coefficients of order $n$ bounded by $n!|\lambda|^n$. 
Benfatto

Rigorous RG techniques (Gallavotti, 1985) have been used to approach the 3D problem; one can prove that the theory is order by order finite in the renormalized coupling constants, with the coefficients of order $n$ bounded by $n!|\lambda|^n$.

- The scheme used by Benfatto is suitable for possible non-perturbative treatments of the theory.
Renormalization Group approach

Reference papers

Benfatto

Rigorous RG techniques (Gallavotti, 1985) have been used to approach the 3D problem; one can prove that the theory is order by order finite in the renormalized coupling constants, with the coefficients of order $n$ bounded by $n!|\lambda|^n$.

- The scheme used by Benfatto is suitable for possible non-perturbative treatments of the theory.

Pistolesi, Castellani, Di Castro, Strinati

The problem has been investigated for spatial dimensions $1 < d < 3$, implementing local Ward identities in a RG approach and exploiting a dimensional regularization with $\epsilon = 3 - d$. 
Renormalization Group approach

Reference papers

**Benfatto**

Rigorous RG techniques (Gallavotti, 1985) have been used to approach the 3D problem; one can prove that the theory is order by order finite in the renormalized coupling constants, with the coefficients of order $n$ bounded by $n!|\lambda|^n$.

- The scheme used by Benfatto is suitable for possible non-perturbative treatments of the theory.

**Pistolesi, Castellani, Di Castro, Strinati**

The problem has been investigated for spatial dimensions $1 < d < 3$, implementing local Ward identities in a RG approach and exploiting a dimensional regularization with $\epsilon = 3 - d$.

- Local WI are crucial for the control of the 2D theory at all orders.
Renormalization Group approach

Corrections to WI

In condensed matter systems the possible presence of a lattice induces an ultraviolet momentum cutoff, which explicitly breaks local gauge invariance.

Remark:

Recent works\textsuperscript{a} have shown that in low-dimensional systems of interacting fermions (such as Luttinger liquids or graphene) the extra terms appearing in the Ward identities due to the presence of an ultraviolet momentum cutoff cannot be naively neglected.

\textsuperscript{a} Benfatto, Falco, Mastropietro (2009) and Giuliani, Mastropietro, Porta (2010)

It's possible that observable results would derive from the corrections to WI even for bosonic systems\textsuperscript{6}.

\textsuperscript{6} This corrections can be controlled at all orders by implementing the method developed in the context of fermionic systems (Benfatto, Mastropietro, 2005).
Renormalization Group approach

Corrections to WI

In condensed matter systems the possible presence of a lattice induces an ultraviolet momentum cutoff, which explicitly breaks local gauge invariance.

Remark:

Recent works\(^a\) have shown that in low-dimensional systems of interacting fermions (such as Luttinger liquids or graphene) the extra terms appearing in the Ward identities due to the presence of an ultraviolet momentum cutoff can not be naively neglected.

\(^a\) Benfatto, Falco, Mastropietro (2009) and Giuliani, Mastropietro, Porta (2010)

It’s possible that observable results would derive from the corrections to WI even for bosonic systems\(^6\).

\(^6\) This corrections can be controlled at all orders by implementing the method developed in the context of fermionic systems (Benfatto, Mastropietro, 2005).
Outline

1 Motivations
2 Known results and open problems
3 Sketch of the strategy
4 Perspectives

Serena Cenatiempo
Low-dimensional interacting bosons
The program

We fix a priori the condensate density $\rho > 0$ and show that one can fix the chemical potential $\nu = \nu(\lambda, \rho)$ so that

$$S_{\text{interacting}}(x) \xrightarrow{x \to \infty} \rho$$

It is natural to represent the field $\varphi_{x}^{\pm}$ as the sum of independent fields:

$$\varphi_{x}^{\pm} = \xi^{\pm} + \psi_{x}^{\pm}, \quad \text{with} \quad \langle \xi^{-} \xi^{+} \rangle = \rho$$

$$Z = \int P(d\xi) \int P(\psi) e^{-V(\xi + \psi)}$$

Free energy density

$$\mathcal{W}(\xi) = -\frac{1}{|\Lambda|} \log \int e^{-V(\xi + \psi)} P(d\psi)$$
Since we suppose that the Bogoliubov approximation is correct in the limit $\beta, L \to \infty$, we expect that the effective potential $V_\xi(\psi)$ would have a “mexican hat” structure.

The RG treatment is simplified if we change the basic fields:

$$\psi^\pm = \frac{1}{\sqrt{2\rho}} (\psi^x \pm i\psi^t)$$
Since we suppose that the Bogoliubov approximation is correct in the limit $\beta, L \to \infty$, we expect that the effective potential $V_\xi(\psi)$ would have a “mexican hat” structure.

The RG treatment is simplified if we change the basic fields:

$$\psi^\pm = \frac{1}{\sqrt{2\rho}} (\psi^x_1 \pm i \psi^x_t)$$

**Essential choice:** the gaussian measure around which perturbation theory is performed corresponds to Bogoliubov approximation.

$$e^{-\Lambda} V_{\text{eff}}(\xi, \phi) = \int P_B(d\psi) e^{\tilde{V}(0)}(\xi + \psi + \phi)$$

$$P_B(d\psi) = P(d\psi) e^{-\lambda \tilde{V}(0)} \int dx (\psi^-_x \xi^+ + \psi^+_x \xi^-)^2$$
Multiscale analysis

Effective potentials

We want to study an infrared problem; we shall consider a simplified model introducing a smooth function $t_0(k)$, which impose an ultraviolet cutoff on scale of the potential $p_0$: 

$$
\int P B (d \psi) \psi \sigma_1 x \psi \sigma_2 y \sigma_i = \rho \left( k^2 - 4 \lambda \tilde{v}(0) \rho \right)
$$

where $\sigma_1$ and $\sigma_2$ are components of the effective potential.
Multiscale analysis

Effective potentials

We want to study an infrared problem; we shall consider a simplified model introducing a smooth function $t_0(k)$, which impose an ultraviolet cutoff on scale of the potential $p_0$:

$$ \tilde{C}_{\sigma_1 \sigma_2}(x - y) = \int P_B(d\psi) \psi_x^{\sigma_1} \psi_y^{\sigma_2} \quad \sigma_i = l, t $$

$$ \tilde{C}_{\sigma_1 \sigma_2}^{\leq 0}(x - y) = \frac{1}{(2\pi)^4} \int dk \: e^{-ik(x-y)} \: t_0(k) \: \tilde{G}_0^{-1}(k)_{\sigma_1 \sigma_2} $$

$$ \tilde{G}_0(k) = \rho \begin{pmatrix} \frac{k^2}{2m} + 4\lambda \tilde{v}(0) \rho \: t_0(k) & ik_0 \\ -ik_0 & -\frac{k^2}{2m} \end{pmatrix} $$
We want to study an infrared problem; we shall consider a simplified model introducing a smooth function $t_0(k)$, which impose an ultraviolet cutoff on scale of the potential $p_0$:

\[
\tilde{C}_{\sigma_1\sigma_2}(x - y) = \int P_B(d\psi) \psi_x^{\sigma_1} \psi_y^{\sigma_2} \quad \sigma_i = l, t
\]

\[
\tilde{C}^{\leq 0}_{\sigma_1\sigma_2}(x - y) = \frac{1}{(2\pi)^4} \int dk e^{-ik(x-y)} t_0(k) \tilde{G}^{-1}_0(k)_{\sigma_1\sigma_2}
\]

\[
\tilde{G}_0(k) = \rho \begin{pmatrix}
\frac{k^2}{2m} + 4\lambda \tilde{v}(0) \rho t_0(k) & ik_0 \\
-i k_0 & -\frac{k^2}{2m}
\end{pmatrix}
\]

We define a family of effective potentials based on a multiscale decomposition of the covariance: $T_h(k) = t_0(\gamma^{-h}k) - t_0(\gamma^{-h+1}k)$

\[
\tilde{C}_h^{\sigma_1\sigma_2}(x - y) = \frac{1}{(2\pi)^4} \int dk e^{-ik(x-y)} T_h(k) \tilde{G}^{-1}_0(k)_{\sigma_1\sigma_2}
\]
Multiscale analysis

Effective potentials

We want to study an infrared problem; we shall consider a simplified model introducing a smooth function $t_0(k)$, which impose an ultraviolet cutoff on scale of the potential $p_0$:

$$\tilde{C}_{\sigma_1\sigma_2}(x - y) = \int P_B(d\psi)\psi_x^{\sigma_1}\psi_y^{\sigma_2} \quad \sigma_i = l, t$$

$$\tilde{C}_{\leq 0}^{\sigma_1\sigma_2}(x - y) = \frac{1}{(2\pi)^4} \int dk e^{-ik(x-y)} t_0(k) \tilde{G}_0^{-1}(k)_{\sigma_1\sigma_2}$$

$$\tilde{G}_0(k) = \rho \left( \begin{array}{cc} \frac{k^2}{2m} + 4\lambda \tilde{v}(0)\rho t_0(k) & ik_0 \\ -ik_0 & -\frac{k^2}{2m} \end{array} \right)$$

We define a family of effective potentials based on a multiscale decomposition of the covariance: $T_h(k) = t_0(\gamma^{-h}k) - t_0(\gamma^{-h+1}k)$

$$\tilde{C}_h^{\sigma_1\sigma_2}(x - y) = \frac{1}{(2\pi)^4} \int dk e^{-ik(x-y)} T_h(k) \tilde{G}_0^{-1}(k)_{\sigma_1\sigma_2}$$
Running constant’s flow

Global WI

By the global gauge transformation $\psi_{x}^{t,l} \rightarrow e^{i\alpha} \psi_{x}^{t,l}$ we obtain which local monomials appear in the potential on scale $h$ ($\psi_{x}^{t} \rightarrow$ plain line; $\psi_{x}^{l} \rightarrow$ dashed line):

$$\mathcal{L}V^{(h)} = \lambda_{h} + \mu_{h} + \gamma^{2h}\nu_{h}$$

and two exact identities which play a very important role in the discussion of the flow of the running coupling constants: $\mu_{h} = -\sqrt{2}Z_{h}(0)$; $\lambda_{h} = Z_{h}/4$.

Deriving the effective potential respect to $\xi^{\pm}$ we find the renormalization condition:

$$\gamma^{2h}\nu_{h} \mathop{\longrightarrow}\limits_{h \rightarrow -\infty} 0$$
Multiscale analysis

**Step by step**

- We integrate iteratively the fields of decreasing scale; the effective potential on scale $h$ is defined by

$$e^{-V_h(\psi)} = \int P_B^{(h+1)}(d\psi^{(h+1)}) e^{-V_{h+1}(\psi + \psi^{(h+1)})}$$

- At each step we localize the relevant or marginal terms and we put all the quadratic terms in the free integration.

- The result of this iterative integration is exactly written as a series in the running coupling constants

$$\vec{r}_h = \vec{r}_h(\vec{r}_{h+1}, \ldots, \vec{r}_0)$$

**Remark:** Each step of the multiscale integration corresponds to a very large resummation of Feynman graphs.
Multiscale analysis

Step by step

- We integrate iteratively the fields of decreasing scale; the effective potential on scale $h$ is defined by

$$e^{-V_h(\psi)} = \int P_B^{(h+1)}(d\psi^{(h+1)}) e^{-V_{h+1}(\psi+\psi^{(h+1)})}$$

- At each step we localize the relevant or marginal terms and we put all the quadratic terms in the free integration.

- The result of this iterative integration is exactly written as a series in the running coupling constants

$$\vec{r}_h = \vec{r}_h(\vec{r}_{h+1}, \ldots, \vec{r}_0)$$

Remark: Each step of the multiscale integration corresponds to a very large resummation of Feynman graphs.
Running coupling constants’ flow

Results

After $|h|$ integrations one gets an integral similar to the initial one with renormalized coupling constants $\lambda_h$, $\mu_h$, $\nu_h$ associated to energy scales $\gamma^{2h} p_0^2$ and a renormalized covariance:

$$\tilde{G}^{\leq h}(k) = \rho \left( \frac{k^2}{2m} + 4 \frac{p_0^2}{2m} \tilde{Z}_h(k) - i k_0 \tilde{E}_h(k) \right) - \left( \frac{k^2}{2m} + 4 \tilde{B}_h(k) \right) \frac{2m}{p_0^2} k_0^2 + 4 \tilde{A}_h(k) \frac{k^2}{2m}$$

$$\tilde{r}_0 \equiv (\lambda_0, \mu_0, \nu_0; \tilde{Z}_0, \tilde{A}_0, \tilde{B}_0, \tilde{E}_0) \equiv \left( \frac{\epsilon}{4}, -\epsilon \sqrt{2}, \frac{\nu^0}{2} \frac{2m}{p_0^2}; \epsilon t_0(k), 0, 0, 1 \right)^7$$

The flow is asymptotically free except for some 2-point functions which renormalize some components of the covariance of the Bogoliubov measure:

$$\tilde{r}_h \equiv \left( \lambda_h \sim \frac{1}{|h|}; \mu_h \sim \frac{1}{|h|}; |\nu_h| \leq \tilde{\nu}; \tilde{Z}_h \sim \frac{1}{|h|}; \tilde{A}_h \sim c_1; \tilde{B}_h \sim c_2; \tilde{E}_h \sim \frac{1}{|h|} \right)$$

$^7 \epsilon = \lambda \tilde{\nu}(0) \rho(2m)/p_0^2$ and $\nu^0$ is the correction to the chemical potential due to the quartic and cubic interaction terms, that is $\nu = -2\lambda \tilde{\nu}(0) \rho + \nu^0$.
Running coupling constants’ flow

After \(|h|\) integrations one gets an integral similar to the initial one with renormalized coupling constants \(\lambda_h, \mu_h, \nu_h\) associated to energy scales \(\gamma^{2h} p_0^2\) and a renormalized covariance:

\[
\tilde{G}^{\leq h}(k) = \rho \left( \begin{array}{cc}
\frac{k^2}{2m} + 4 \frac{p_0^2}{2m} \tilde{Z}_h(k) & ik_0 \tilde{E}_h(k) \\
-ik_0 \tilde{E}_h(k) & - \left( \frac{k^2}{2m} + 4 \tilde{B}_h(k) \right) \frac{2m}{p_0^2} k_0^2 + 4 \tilde{A}_h(k) \frac{k^2}{2m} \end{array} \right)
\]

\[
\vec{r}_0 \equiv (\lambda_0, \mu_0, \nu_0; \tilde{Z}_0, \tilde{A}_0, \tilde{B}_0, \tilde{E}_0) \equiv \left( \frac{\varepsilon}{4}, -\varepsilon \sqrt{2}, \frac{\nu^0}{2} \frac{2m}{p_0^2}; \varepsilon t_0(k), 0, 0, 1 \right)^7
\]

The flow is asymptotically free except for some 2-point functions which renormalize some components of the covariance of the Bogoliubov measure:

\[
\vec{r}_h \equiv \left( \lambda_h \sim \frac{1}{|h|}; \mu_h \sim \frac{1}{|h|}; |\nu_h| \leq \bar{\nu}; \tilde{Z}_h \sim \frac{1}{|h|}; \tilde{A}_h \sim c_1; \tilde{B}_h \sim c_2; \tilde{E}_h \sim \frac{1}{|h|} \right)
\]

\[\varepsilon = \lambda \bar{\nu}(0) \rho(2m)/p_0^2\] and \(\nu^0\) is the correction to the chemical potential due to the quartic and cubic interaction terms, that is \(\nu = -2\lambda \bar{\nu}(0) \rho + \nu^0\)
Perspectives

*Short and long run*

- To develop the strategy presented for the 3D case to study the order by order estimates for the two dimensional Bose gas. *(the analysis of Pistolesi et al. predicts a non trivial fixed point and no anomalous dimensions)*
  - To control the corrections to WI at all orders.

- To develop the same ideas to handle the study of interacting bosons in 1D, to get a systematic analysis of the correlation functions.

- To combine Renormalization Group ideas with the stationary phase approximations techniques\(^8\) to solve the large field problem.

---

\(^8\) Balaban, Feldman, Knörrer, Trubowitz (2009)
Perspectives

*Short and long run*

- To develop the strategy presented for the 3D case to study the order by order estimates for the two dimensional Bose gas. (the analysis of Pistolesi et al. predicts a non trivial fixed point and no anomalous dimensions)
  - To control the corrections to WI at all orders.

- To develop the same ideas to handle the study of interacting bosons in 1D, to get a systematic analysis of the correlation functions.

- To combine Renormalization Group ideas with the stationary phase approximations techniques\(^8\) to solve the large field problem.

---

\(^8\) Balaban, Feldman, Knörrer, Trubowitz (2009)
Perspectives

*Short and long run*

- To develop the strategy presented for the 3D case to study the order by order estimates for the two dimensional Bose gas. *(the analysis of Pistolesi et al. predicts a non trivial fixed point and no anomalous dimensions)*
  - To control the corrections to WI at all orders.

- To develop the same ideas to handle the study of interacting bosons in 1D, to get a systematic analysis of the correlation functions.

- To combine Renormalization Group ideas with the stationary phase approximations techniques\(^8\) to solve the large field problem.

---

\(^8\) Balaban, Feldman, Knörrer, Trubowitz (2009)