An effective model for graphene with long range interactions

PhD Research Project
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Introduction

Graphene is a newly discovered material, and it consists in a monoatomic layer of graphite; therefore, it can be considered a truly bidimensional system. It has been isolated for the first time in 2005, see [1], and it is surprisingly stable.

As the experiments have shown, the low energy excitations of graphene behave like "slow" massless Dirac electrons in $2 + 1$ dimensions, [1]; in fact, their energy dispersion is linear near the Fermi surface, which in this system consists in two isolated points (Dirac points), see figure 1, and experimental observations show that the velocity $v$ of the electrons is much smaller than the speed of light $c$ ($v \sim \frac{c}{100}$). Pictorially, we can say that the physics of graphene is described by a “slow” relativity, [2].

The relativistic behavior of graphene has been predicted by many far before its experimental discovery, see [3], [4], [5] for instance; it is a consequence of the hexagonal structure of its crystalline lattice. This peculiar feature has deep consequences on the physical properties of the material. For instance, graphene is insensitive to the localization effects usually induced by disorder, which translates into the fact that it shows a very high electronic mobility, the lowest known resistivity, and very long electronic mean free paths (the motion of the electrons is ballistic on the micrometer scale, see [6]); this effect is understood by many as an evidence of the “relativistic” nature of graphene, since it is well known that Dirac electrons do not localize. Other interesting physical properties are the presence of an anomalous integer quantum Hall effect, a nonzero minimal value of the conductivity (equal to $\frac{e^2}{17}$, value reached in presence of a vanishing density of charge carriers), and an anomalous dependence of the cyclotron mass of the charge carriers on the electronic density. These and other remarkable physical properties make graphene very interesting for technological applications, [7].
Figure 1: A schematic representation of the electronic energy dispersion of graphene. The Fermi surface is given by the intersection of the “cones” (the so-called Dirac cones) with the horizontal plane; only two of the six Dirac points are independent, since the others are out of the first Brillouin zone.

Many efforts are currently devoted to the theoretical analysis of this physical system, in particular to the effect of electron-electron interactions, [5], [8], [9], [10], [11]. From a theoretical point of view a good candidate to describe graphene is the two dimensional Hubbard model on the honeycomb lattice. The Hamiltonian of this model is given by the sum of two parts: the first takes into account the hopping of the electrons from their sites to the nearest neighbours, while the second contains the two-body interaction between the electrons. In the case of short range interactions (i.e. that involve only neighbouring sites or more generally electrons separated by a finite number of sites) it is believed that, on the basis of lowest order computations, contrary to what happens in the 2d Hubbard model on the square lattice, the interaction does not produce any quantum instability in the zero temperature limit, [12], [10], [13].

Of course it remained open the possibility of instabilities at weak coupling due to nonperturbative effects; this has been ruled out by the analysis of Giuliani and Mastropietro, see [14], [15]. They proved that at half-filling (that is when each site of the lattice is occupied on average by one electron) and for sufficiently small coupling $U$ the perturbative series defining the ground state correlations is convergent, excluding the presence of magnetic or superconducting instabilities or the formation of a mass gap. All the effect of the interaction is contained in a dressing of the Fermi velocity and of the wave function renormalization of the quasi electrons, which differ of $O(U)$ from their bare counterpart. Moreover, despite the lattice is not invariant under $\frac{\pi}{2}$ rotations, the effective electronic dispersion (Dirac cone) shows a cylindrical symmetry in $\vec{k}$ space.

The proof is based on a rigorous formulation of the Wilsonian renormalization group, introduced by Gallavotti in the 80’s to renormalize the $\varphi^4$ field theory in four dimensions, [16]; these techniques have shown to be very useful in the study of fermionic field theories: for instance, we mention the
proof of the Luttinger liquid behavior for one dimensional systems, [17], the proof of the Fermi liquid behavior of the 2d Hubbard model on square lattice at small filling and up to an exponentially small temperature, [18], and more recently the proof of scaling relations between the critical indices of a wide class of quantum spin chains, [19].

Results

My PhD thesis focuses on graphene at half – filling with long range interaction. In this case the situation is much less clear with respect to the short range case; first of all, one should distinguish between instantaneous long range interactions, the “true” Coulomb interaction, and retarded ones, where the photon has a finite velocity $c$.

**Instantaneous case.** In the instantaneous case it has been proposed, on the basis of lowest order perturbation theory, that the Fermi velocity diverges as $e^2 \log |k - p_F^e|$, where $p_F^e$ with $\omega = \pm$ is one of the two points forming the Fermi surface and $e$ is the electric charge, while the wave function renormalization is close to its unperturbed value $Z = 1$ up to $O(e^2)$, [20], [9], [21], [22], [23]; of course, the calculations are made under the assumption that the theory is renormalizable. These results, in particular the logarithmic growth of the Fermi velocity, are very debated; in fact, an eventual proliferation of $\log$s at higher orders could produce an anomalous scaling of the Fermi velocity, as the identity $|k| e^2 \approx \sum_{n \geq 0} \left( e^2 \log |k|^n \right)^n$ suggests.

We started studying the instantaneous case on the honeycomb lattice, and we confirmed these lowest order predictions; however, from the dimensional analysis of higher order Feynman graphs we cannot conclude that the theory is renormalizable. In fact, it seems that at higher orders the graphs scale differently in the infrared cutoff, essentially because the bosonic propagator does not depend on $k_0$, the Matsubara frequency. Hence, it is unclear if one should look for subtle cancellations in the Feynman graphs, or if the theory is just non renormalizable.

**Retarded case.** In the case of retarded long range interaction an effective model for graphene at half – filling has been proposed by Guinea, González and Vozmediano in [5]; it essentially consists in a $3+1$ quantum electrodynamics in presence of an ultraviolet cutoff were the massless fermions are constrained to live on a plane and have bare Fermi velocity $v < c$. This model has been studied in [5] at second order via renormalization group methods using dimensional regularization; the main claim is that the interaction produces an anomalous scaling of the correlations, e.g. that the electronic interacting two point correlation function $g^{int}$ behaves as, in $k$ space,

$$g^{int}(k) \simeq \frac{1}{|k|^\eta} g^{bare}(k), \quad k = (k_0, \vec{k}),$$

where $\eta = O(e^2) > 0$ and $g^{bare}$ is the electronic non interacting two point function. Moreover, always relying on second order computations and dimensional regularization, the authors argued that the fermionic effective velocity flows to the speed of light $c$, that the electric charge is not renormalized, and that the long range interaction is unscreened.

Our work consists in a study of this model at all orders in perturbation theory; the main motivation of the analysis is that there is a debate on the presence of a critical exponent $\eta$ in the decay of the correlations of graphene, mainly because, as for the Fermi velocity in the instantaneous case, from lowest order computations it is not clear how to distinguish $|k|^\eta$ from $1 + \eta \log k$, [5],
Formula (1), if true, would imply that the interacting system is not a Fermi liquid, but rather a Luttinger liquid, [25]; this behavior is quite common in one dimensional fermionic systems, but it has never been rigorously established in two dimensional systems. Luttinger liquids are very interesting in physics; for instance, a Luttinger liquid phase, [26], has been proposed as an explanation of some properties of high temperature superconductors.

Our results are that, at all orders in perturbation theory,

1. formula (1) is correct, with \( \eta \) given by a formal perturbative series in the renormalized charge;
2. the flow of the renormalized charge is bounded, that is the effective charge is \( e(k) = e + O(e^3) \) where \( e \) is the bare one;
3. the effective fermionic velocity flows in the infrared to \( c \) with an anomalous exponent given by a formal perturbative series in the renormalized charge, that is if the bare velocity is \( v = c(1 - d) \) with \( d \) small, then the effective velocity is \( v(k) \simeq c(1 - d|k|^\gamma) \) with \( \gamma = O(e^2) > 0 \);
4. there is no mass generation for the fermionic field.

To prove result 2 we had to show the so–called vanishing of the beta function of the electric charge. This property can be checked quite easily at one loop in perturbation theory, but to prove it at all orders it has been crucial to use Ward identities, that in presence of cutoffs are different from the formal ones (obtained neglecting the cutoffs); it follows that the correction terms due to the presence of the ultraviolet cutoff can be controlled, and they give an additive contribution \( O(e^3) \) to the effective charge. We stress that the analysis at all orders of the flow of the electric charge is not just a mathematical subtlety; it would be enough to have a single order different from zero to make the flow of the charge unbounded or vanishing, while what we are saying here is that the effective electric constant remains close to its bare value \( e \).

Result 3 implies that Lorentz symmetry, which is broken in the bare theory since \( v \neq c \), is dynamically restored, that is QED is a fixed point of renormalization group. Finally, result 4 implies that there is no mass gap, and it is a consequence of the (non relativistic) symmetries of the model.

For what concerns the unscreening of the interaction, the result obtained in [5] through dimensional regularization is not reproducible in our rigorous framework, where we have to impose it “by hand” through a bare mass counterterm. In fact, the presence of the ultraviolet cutoff, which mimics a space – time symmetric lattice, generates a mass for the bosonic field, which is unphysical for our purposes since the long range interactions are widely believed to be unscreened in graphene. This is confirmed by a second order computation in the instantaneous case on the physical lattice, the honeycomb one; but again, in this case we cannot say anything on higher orders.

**Perspectives**

Our claims have to be understood order by order in perturbation theory, that is at the moment we do not control the convergence of the perturbative series; at present time we can only say that the \( n \)-th order of the series in the renormalized coupling constant is bounded by \( n!e^n \), which is clearly not enough to perform the sum. Convergence issues will be object of future studies; we plan to attack the problem using the well known determinant bounds, which is the standard technique...
adopted to prove convergence for fermionic systems. Roughly speaking, one exploits the $-1$ arising in the anticommutation of fermionic fields to prove that the $n$-th order of the series, given by $O(n!)$ addends, reconstructs a determinant of an $n \times n$ matrix, which is estimated by $\text{const}^n$; but in our case the proof is not a straightforward application of the usual methods, since one has to deal with both bosonic and fermionic fields. An eventual proof of the convergence of the series would be interesting not only for condensed matter physics, but also from the point of view of quantum field theory, since at present time very few result on convergence of infrared $3+1$ dimensional QED are present in literature (the existing results assume a large mass for the fermionic field - much larger than the electric charge, see [27], or a large photon mass, see [28]). But clearly our case is a very special one, being the fermions restricted to a plane; this fact represents a great simplification in the renormalization group analysis, and in particular it is at the origin of the finiteness of the photon wave function renormalization.

Another issue that we are currently investigating is the presence of the lattice; despite its presence is irrelevant in a renormalization group sense, it deserves a nontrivial discussion since it explicitly breaks Lorentz invariance not only because the velocity of the electrons is different from $c$ but also because the cutoff induced by the lattice is asymmetric in $k$, since it involves only the spatial part $\vec{k}$ of $k = (k_0, \vec{k})$. Moreover, in presence of the honeycomb lattice the energy dispersion $\varepsilon(\vec{k})$ of the electrons is only asymptotically linear, as it is evident from figure 1; hence, if one chooses to couple the system with the electromagnetic field $A_k = (A_{k,0}, \vec{A}_k)$ through the minimal substitution

$$k_\mu \rightarrow k_\mu - e c A_{k,\mu} \ , \quad (2)$$

the nonlinearity of $\varepsilon(\vec{k})$ implies that the interacting Hamiltonian is not linear in $A_{k,\mu}$, which makes the model untreatable from an analytical point of view. One can try to linearize the Hamiltonian in $A_{k,\mu}$; some preliminary analysis suggest that the resulting model can be studied in a way analogous to that followed in the case of QED with symmetric cutoff and different velocities. In particular, there seems to be mass generation for the components $A_{k,1}, A_{k,2}$ of the field, while $A_{k,0}$ can be proved to be massless at all energy scales by using a Ward identity; therefore, even in this case it is likely that the unscreening of the long range interaction has to be imposed by hand with a mass counterterm.

To conclude, it would be very interesting to investigate further the issue of renormalizability of the instantaneous case on the honeycomb lattice. As mentioned in the previous section, in this case dimensional analysis alone is not enough to prove the renormalizability of the theory; to accomplish this it is necessary to improve the bounds on the coefficients of the perturbative series by looking at eventual cancellations in the Feynman graphs. From some preliminary computations it seems that, at least for some classes of diagrams, the dimensional bounds can be improved using momentum conservation and symmetry arguments; but at the moment it is not clear at all if these cancellations can be identified in a systematic way in all the graphs.

References


